Matrix Representation of Double Layered Non–Cyclic Fuzzy Graph

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Abstract

Matrix representation of fuzzy relations has become an important tool in the field of science, medical diagnosis and engineering. In this paper we have introduced the double layered non-cyclic fuzzy graph in matrix representation with respect to the vertices. We have also discussed some of its properties and explained it with numerical examples.

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1 Introduction

Matrices are one of the most powerful tools in mathematics which perform calculations in a very simple and compact form. The classical mathematics fails to solve problems that deals with uncertainties occur in a vague environment. The concept of fuzzy sets was introduced by Zadeh in 1965 \[8\]. Rosenfeld in 1975 introduced fuzzy graph theory \[9\]. Thomson introduced the concept of fuzzy matrix in 1977 and he discussed
the convergence of the powers of a fuzzy matrix [5]. As an extension of
Boolean matrices Kim and Roush developed the theory of fuzzy matrices
[6]. Ragab and Emam discussed the determinant and adjoint of a fuzzy
square matrix [7]. T. Pathinathan and J. Jesintha Roseline defined the
double layered fuzzy graph in 2014 [1]. In the same year they introduced
the matrix representation of double layered fuzzy graph and also studied
its properties [2]. In this paper we have represented the 3D - structure
of non - cyclic fuzzy graph in the form of fuzzy matrix and also studied
some of its properties with an example.

2 Preliminaries

2.1 Definition

A fuzzy graph G is a pair of function $G : (\sigma, \mu)$ where $\sigma$ is a fuzzy
subset of non-empty set $S$ and $\mu$ is a symmetric fuzzy relation on $\sigma$. The
underlying crisp graph of $G : (\sigma, \mu)$ is denoted by $G^* : (\sigma^*, \mu^*)$.

2.2 Definition

Let $G : (\sigma, \mu)$ be a fuzzy graph with the underlying crisp graph $G^* :
(\sigma^*, \mu^*)$. The pair $DL(G) : (\sigma_{DL}, \mu_{DL})$ is defined as follows. The node
set of $DL(G)$ be $\sigma^* \cup \mu^*$. The fuzzy subset $\sigma_{DL}$ is defined as
$\sigma_{DL} = \begin{cases} 
\sigma(u) & \text{if } u \in \sigma^* \\
\mu(uv) & \text{if } uv \in \mu^*
\end{cases}$
The fuzzy relation $\mu_{DL}$ on $V \cup E$ is defined as
$\mu_{DL} = \begin{cases} 
\mu(uv) & \text{if } u, v \in \sigma^*. \\
\mu(e_i) \land \mu(e_j) & \text{if the edge } e_i \text{ and } e_j \text{ have node in common between them.} \\
\mu(u_i) \land \mu(e_i) & \text{if } u_i \in \sigma^* \text{ and } e_i \in \mu^* \text{ and each } e_i \text{ incident with single } u_i \text{ either} \\
& \text{clockwise or anticlockwise.} \\
0 & \text{otherwise.}
\end{cases}$
By definition, $\mu_{DL}(u, v) \leq \sigma_{DL}(u) \land \sigma_{DL}(v) \forall u, v \in \sigma^* \cup \mu^*$. Here $\mu_{DL}$
is a fuzzy relation on the fuzzy subset $\sigma_{DL}$. Hence the pair $DL(G) : 
(\sigma_{DL}, \mu_{DL})$ is a fuzzy graph and is termed as Double Layered Fuzzy
Graph.

2.3 Definition

Let $G : (\sigma, \mu)$ be a fuzzy graph with the underlying crisp graph $G^* :
(\sigma^*, \mu^*)$. The pair $NCDL(G) : (\sigma_{NCDL}, \mu_{NCDL})$ is defined as follows.
The vertex set of \( NCDL(G) \) be \( \sigma^* \cup \mu^* \).
The fuzzy subset \( \sigma_{NCDL} \) is defined as
\[
\sigma_{NCDL} = \sigma(u) \text{ if } u \in \sigma^*.
\]
\[
\sigma_{NCDL} = \mu(uv) \text{ if } uv \in \mu^*.
\]
and the fuzzy relation \( \mu_{NCDL} \) on \( \sigma^* \cup \mu^* \) is defined as

1. \( \mu_{NCDL} = \mu(u_iu_j) = e_i \text{ if } e_i \text{ is incident with } \sigma(u_i) \text{ and } \sigma(u_j), \) i.e. there is only one edge incident with the corresponding vertex if \( u_i, u_j \in \sigma^* \) and \( e_i \in \mu^* \).

2. \( \mu_{NCDL} = \mu(u_iu_j) \wedge \mu(u_iu_k) = e_i \wedge e_j \text{ if } e_i \text{ is incident with } \sigma(u_i) \) and \( \sigma(u_j), e_j \text{ is incident with } \sigma(u_i) \text{ and } \sigma(u_k) \) i.e. there exist more than one edge incident with the corresponding vertex and if \( u_i, u_j, u_k \in \sigma^* \) and \( e_i, e_j, e_k \in \mu^* \).

3. \( \mu_{NCDL} = \mu(e_i) \wedge \mu(e_j) \) if the edge \( e_i \) and \( e_j \) have a node in common between them.

4. \( \mu_{NCDL} = \sigma(u_i) \wedge \mu(e_i) \) if \( u_i \in \sigma^* \) and \( e_i \in \mu^* \), each \( e_i \) is incident with corresponding single \( u_i \).

5. \( \mu_{NCDL} = 0 \) otherwise.

By definition, \( \mu_{NCDL}(u,v) \leq \sigma_{NCDL}(u) \wedge \sigma_{NCDL}(v) \forall u, v \in \sigma^* \cup \mu^* \).
Here \( \mu_{NCDL} \) is a fuzzy relation on the fuzzy subset \( \sigma_{NCDL} \). Hence the pair \( NCDL(G) : (\sigma_{NCDL}, \mu_{NCDL}) \) is a fuzzy graph and is termed as Double Layered Non-Cyclic Fuzzy Graph.

3 Matrix Representation of Non-Cyclic Double Layered Fuzzy Graph

Consider a fuzzy graph \( G \) with \( n=3 \) vertices

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The matrix representation for the given fuzzy graph \( G : (\sigma, \mu) \) with respect to its vertices is given below

\[
M_G = \begin{bmatrix}
0.3 & 0.3 & 0 \\
0.3 & 0.4 & 0.4 \\
0.3 & 0.4 & 0.5
\end{bmatrix}
\]

The 3D-Structure for the non-cyclic fuzzy graph \( G : (\sigma, \mu) \) is given by

The matrix representation for the non-cyclic fuzzy graph \( G : (\sigma, \mu) \) is given by

\[
M_{NG} = \begin{bmatrix}
0.3 & 0.3 & 0 & 0 & 0 \\
0.3 & 0.4 & 0.4 & 0 & 0 \\
0.3 & 0.4 & 0.5 & 0 & 0 \\
0.3 & 0 & 0.3 & 0.3 & 0 \\
0.3 & 0 & 0.3 & 0.3 & 0.3 \\
0.3 & 0 & 0.3 & 0.4 & 0.3 \\
0.3 & 0.3 & 0.4 & 0.3 & 0.4
\end{bmatrix}
\]
3.1 Vertex Matrix Representation of Double Layered Non-Cyclic Fuzzy Graphs

For a non-cyclic fuzzy graph $G = (\sigma, \mu)$ with the fuzzy relation $\mu$ to be reflexive and symmetric, the edge matrix $M_{\text{NCDLG}}\sigma$ is defined as follows:

$$M_{\text{NCDLG}}\sigma = \begin{cases} 
\mu(e_i) \land \mu(e_j) & \text{if } i \neq j, \\
\mu(e_i) & \text{if } i = j, \\
\mu(u_i) \land \mu(e_i) & \text{each } e_i \text{ is incident with single } u_i, \\
0 & \text{otherwise}.
\end{cases}$$

**Example 3.1** For Figure 2, the vertex matrix representation if $e_i \in \sigma^*_\text{NCDL}$ is given by

$$\begin{pmatrix}
0 & 0.3 & 0.3 \\
0.3 & 0 & 0 \\
0.3 & 0 & 0.4
\end{pmatrix}$$

Thus the matrix representation of double layered non-cyclic fuzzy graph becomes

$$M_{\text{NCDLG}}\sigma = \begin{pmatrix} M_G & NCD_G\
NCD_G & NCD_G \end{pmatrix}$$

4 Theoretical Concepts

4.1 Theorem

Trace $(M_{\text{NCDLG}}\sigma) = \text{Order (G)} + \text{Order}_{\text{NCD}}(G)$

Proof: Trace $(M_{\text{NCDLG}}\sigma)$ = Sum of the diagonal entries in $M_{\text{NCDLG}}\sigma$.

$$= \sum_{i=1}^{n} \mu_{\text{NCDLG}(G)}(v_i, v_i) = \sum_{e_i \in \sigma^*} \sigma_G(e_i) + \sum_{e_i \in \mu^*} \sigma_{\text{NCDLG}}(e_i)$$

$$= \text{Order (G)} + \text{Order}_{\text{NCD}}(G).$$

4.2 Theorem

$M_{\text{NCDLG}}\sigma$ is a symmetric matrix

Proof: By the definition of $M_{\text{NCDLG}}\sigma$ it is clear that the relation $\mu$ is a symmetric relation.

Hence $(M_{\text{NCDLG}}\sigma)_{i,j} = \mu(u_i, u_j)$

$$= \mu(u_j, u_i) \cdot \mu \text{ is symmetric}$$
Further from the representation of double layered non-cyclic fuzzy graph
We have $M_{NCDLG} = \left( \begin{array}{cc} M_G & NCD_G \\ NCD_G & NCD_G \end{array} \right)$
and the transpose of this matrix is

$$M^T_{NCDLG} = \left( \begin{array}{cc} M_G & NCD_G \\ NCD_G & NCD_G \end{array} \right)$$

$$\Rightarrow M_{NCDLG} = M^T_{NCDLG}$$
Thus the matrix $M_{NCDLG}$ is symmetric.

### 4.3 Theorem

The sum of all entries in $M_{NCDLG}$ except the diagonal elements is

$$2 \text{ size}(G) + 2 \sum_{i,j=1}^{n} \mu(u_i) \land \mu(e_j) + 2 \sum_{i=1}^{n} \mu(u_i) \land \mu(e_i).$$

Proof: The sum of all the elements except the diagonal elements is

$$\sum_{i,j=1}^{n} \mu(u_i, u_j) \text{ if } u_i \in \sigma^*, e_j \in \sigma_{DL}^* \text{ then we have}$$

$$\sum_{i,j=1}^{n} \left( \sum_{i,j=1}^{n} (M_G)_{ij} + \sum_{i,j=1}^{n} (NCD_G)_{ij} \right)$$

$$= 2 \text{ size}(G) + 2 \sum_{i,j=1}^{n} \mu(e_i) \land \mu(e_j) \text{ and if } u_i \in \mu^*, \text{ then we have}$$

$$\sum_{i=1}^{n} \left( \sum_{i=1}^{n} \mu(u_i) \land \mu(e_i) \right) = 2 \sum_{i=1}^{n} \mu(u_i) \land \mu(e_i).$$

Hence $\sum_{i,j=1}^{n} \mu(u_i, u_j) = 2 \text{ size}(G) + 2 \sum_{i,j=1}^{n} \mu(e_i) \land \mu(e_j) + 2 \sum_{i=1}^{n} \mu(u_i) \land \mu(e_i).$

**Example 4.1** Consider the fuzzy graph G, whose crisp graph $G^*$ is a non-cyclic with $n = 4$ vertices.
Figure 3: Fuzzy Graph $G : (\sigma, \mu)$ whose crisp graph $G^* : (\sigma^*, \mu^*)$ is a non-cyclic.

Figure 4: Double Layered Non-Cyclic Fuzzy Graph NCDL (G) $= (\sigma_{NCDL}, \mu_{NCDL})$. 
Here \( \text{Size} (G) = 1.1 \), $\sum_{i,j}^{n} \mu(e_i) \wedge \mu(e_j) = 0.9$ and $\sum_{i=1}^{n} \mu(u_i) \wedge u_i \neq j$.

\[
\begin{bmatrix}
0.3 & 0 & 0.3 & 0.3 & 0 & 0 & 0 \\
0 & 0.5 & 0.4 & 0 & 0.4 & 0 & 0 \\
0.3 & 0.4 & 0.4 & 0 & 0 & 0.3 & 0 \\
0 & 0.4 & 0 & 0 & 0.4 & 0 & 0.4 \\
0.3 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\
0 & 0 & 0.4 & 0 & 0 & 0.3 & 0.4 \\
\end{bmatrix}
\]

\[
M_{\text{NCDLG}} = \nu_1 u_2 u_3 u_4 e_1 e_2 e_3 e_4 \\
\nu_1 \nu_2 \nu_3 \nu_4 \\
e_1 e_2 e_3 e_4
\]

5 Conclusion

In this paper we have defined the matrix representation of the double layered non-cyclic fuzzy graph and also verified some of its properties along with numerical examples.

References


