COMMON FIXED POINT THEOREMS IN 
$\varepsilon$-CHAINABLE GENERALIZED FUZZY 
METRIC SPACES

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Abstract: We introduce the notion of $\varepsilon$-chainable generalized fuzzy metric spaces. We give some conditions of which six self mappings of $\varepsilon$-chainable generalized fuzzy metric space have a unique common fixed point. Also, we characterized the conditions for three self mappings of $\varepsilon$-chainable generalized fuzzy metric spaces have a unique common fixed point.

Keywords: $\varepsilon$-chainable, Fuzzy metric space, Generalized fuzzy metric spaces, Weakly compatible map.

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1 Introduction

The foundation of fuzzy set theory and fuzzy mathematics were laid down by Zadeh [13] in 1965 by the introduction of the notion of fuzzy sets. The theory of fuzzy sets has vast applications in applied sciences and engineering such as neural network, stability theory, mathematical programming, genetics, nervous systems, image processing, control theory etc. The theory of fixed points is one of the basic tools to handle the physical formulations. This has led to the development and fuzzyfication of several concepts of analysis and topology. In 1975, Kramosil and Michalek introduced the concept of fuzzy metric space by generalizing the concept of a probabilistic a fuzzy metric space to the fuzzy situation. The concept of Kramosil and Michalek [8] of a fuzzy metric space was later modified by George and Veeramani [3] in 1994. In 1988, Grabiec [4], following the concept of Kramosil and Michalek [8], obtained the fuzzy version of Banach’s fixed point theorem. Using the notion of weak commuting property, Sessa [12] improved the commutative conditions in fixed point theorem. Jungck [6] Jungck and Rhoades [7] introduced the concept of compatibility in metric spaces. In 2006, Jungck and Rhoades [7] introduced the concept of weakly compatible maps which was the generalization of the concept of compatible maps. The notion of compatible mapping in fuzzy metric space was introduced by Cho [1] in 1997. Cho et al. introduced the concept of $\varepsilon$- chainable fuzzy metric space and obtained
common fixed point theorems for six weakly compatible mappings of ε chainable fuzzy metric spaces. We introduce the notion of ε-chainable generalized fuzzy metric spaces. We give some conditions of which six self mappings of ε-chainable generalized fuzzy metric space have a unique common fixed point.

2 Preliminaries

Definition 2.1: A binary operation * : [0, 1]x [0, 1] → [0, 1] is a continuous t-norm if it satisfies the following conditions:

1. * is associative and commutative,
2. * is continuous,
3. a * 1 = a for all a ∈ [0,1],
4. a * b ≤ c * d whenever a ≤ c and b ≤ d, for each, a,b, c, d ∈ [0,1].

Two typical examples of continuous t-norms are a * b = ab and a * b = min { a,b } .

Definition 2.2: A triple (X, M, *) is called a generalized fuzzy metric space if X is an arbitrary nonempty set, * is a continuous t-norm and M : X^3 x (0,∞) is a fuzzy set satisfying the following conditions: for all x, y, z, a ∈ X and s, t > 0

1. M(x, y, z, t) > 0,
2. M(x, y, z, t) = 1 ⇔ x = y = z,
3. M(x, y, z, t) = M(p{ x, y, z}t),where p is a permutation function,
4. M(x, y, z, t + s) ≥ M(x, y, a, t) * M(a, z, z, s),
5. M(x, y, z .) : (0, ∞) → [0,1] is a continuous mapping,
6. lim t→∞ M(x, y, z, t) = 1.

Definition 2.3: Let X be a non-empty set and D* be the D*-metric on X. Denote a *b = ab for all a,b ∈ [0,1]. For any t ∈ (0, ∞), define $M(x, y, z, t) = \frac{t}{t + D*(x, y, z)}$ for all x, y, z ∈ X, then (X, M, *) is a generalized fuzzy metric space. It is called as M-fuzzy metric induced by D*-metric. Thus every D*-metric induces a generalized fuzzy metric.

Definition 2.4: Let (X, M, *) be a generalized fuzzy metric space and \{x_n\} be a sequence in X

i. A sequence \{x_n\} in X is said to be converge to a point x ∈ X, if $\lim_{n \to \infty} M(x, x_n, t) = 1$ for all t > 0.

ii. A sequence \{x_n\} in X is called Cauchy sequence if $\lim_{n \to \infty} M(x_{n+p}, x_{n+p}, x_n, t) = 1$ for all t > 0 and p > 0.

iii. A generalized fuzzy metric space, in which every Cauchy sequence is convergent, is said to be complete.
Lemma 2.5: Let \((X, M, \ast)\) be a generalized fuzzy metric space. Then, for every \(t > 0\) and for every \(x, y \in X\). We have \(M(x, y, t) = M(y, x, t)\).

Lemma 2.6: Let \((X, M, \ast)\) be a generalized fuzzy metric space. Then, \(M(x, y, z, t)\) is non-decreasing with respect to \(t\), for all \(x, y, z \in X\).

Lemma 2.7: Let \((X, M, \ast)\) be a generalized fuzzy metric space with condition (FM-6). If there exists \(k \in (0, 1)\) such that \(M(x, y, z, kt) \geq M(x, y, z, t)\), for all \(x, y, z \in X\) and \(t > 0\), then \(x = y = z\).

Lemma 2.8: Let \(\{x_n\}\) be a sequence in a generalized fuzzy metric space \((X, M, \ast)\) with condition (FM-6). If there exists \(k \in (0, 1)\) such that \(M(x_n, x_{n+1}, x_{n+1}, kt) \geq M(x_{n-1}, x_n, x_n, t)\), for all \(t > 0\), and \(n = 1, 2, \ldots\), then \(\{x_n\}\) is a Cauchy sequence.

Definition 2.9: Let \(A\) and \(S\) be mappings from a generalized fuzzy metric space \((X, M, \ast)\) into itself. Then, the mappings are said to be compatible, if
\[
\lim_{n \to \infty} M(ASx_n, SAx_n, SAx_n, t) = 1 \quad \text{for all} \ t > 0 \quad \text{whenever} \ \{x_n\} \ \text{is a sequence in} \ X \ \text{such that} \ lim_{n \to \infty} Ax_n = lim_{n \to \infty} Sx_n = x \in X.
\]

Definition 2.10: Two self mapping \(A\) and \(S\) of a generalized fuzzy metric space \((X, M, \ast)\) are said to be weakly compatible, if \(ASx = SAx\) whenever \(Ax = Sx\) for some \(x \in X\).

Definition 2.11: A finite sequence \(x = x_0, x_1, x_2, \ldots x_n = y\) in a generalized fuzzy metric space \((X, M, \ast)\) is called \(\varepsilon\)-chain from \(x\) to \(y\) if there exists \(\varepsilon > 0\) such that
\(M(x_i, x_{i-1}, x_{i-1}, t) > 1 - \varepsilon\) for all \(t > 0\) and \(i = 1, 2, \ldots, n\).
A fuzzy metric space \((X, M, \ast)\) is called \(\varepsilon\)-chainable if there exists an \(\varepsilon\)-chain from \(x\) to \(y\), for any \(x, y \in X\).

2 Main Results

Let \((X, M, \ast)\) be a complete \(\varepsilon\)-chainable, generalized fuzzy metric space and let \(A, B, C, P, Q, R\) be the self mappings of \(X\), satisfying the following conditions:

1. \(A(X) \subset Q(X), B(X) \subset P(X), C(X) \subset R(X)\),
2. The pair \((A, R), (B, Q), (C, P)\) are weakly compatible,
3. There exists a constant \(k \in (0, 1)\) such that for every \(x, y, z \in X\) and \(t > 0\),
\[
M(Ax, By, Cz, t) \geq M(Rx, Qy, Pz, t) \ast M(Ax, By, Cz, t) \ast M(By, Qy, Ax, t) \ast M(Cz, Pz, By, t) \ast M(Ax, Px, By, t) \ast M(Bz, Rx, Cz, t) \ast M(Ax, Qx, Bz, t) \ast M(By, Py, Cy, t) \ast M(Cz, Pz, Az, t)
\]

Then \(A, B, C, P, Q, R\) have a unique common fixed point in \(X\).

Proof:
We can find a Cauchy sequence \(\{y_n\}\) in \(X\) such that
\[ y_{2n-1} = Qx_{2n-1} = Ax_{2n-2}, \\
y_{2n} = Px_{2n} = Bx_{2n-1}, \\
y_{2n+1} = Rx_{2n+1} = Cx_{2n}, \text{for } n = 1, 2, 3, \ldots \]

From completeness, \( y_n \to z \) for some \( z \in X \), and so \( \{Ax_{2n-2}\}, \{Px_{2n}\}, \{Bx_{2n-1}\}, \{Rx_{2n-1}\}, \{Cx_{2n}\} \) also converge to \( z \).

We can show that \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, hence there exists \( z \in X \) such that \( \{x_n\} \) converge to \( z \). Hence, there exists \( u, v, w \in X \) such that 

\[
M(Ru, z, z, t) \geq M(Au, z, z, t) + M(z, z, z, t) + 1 \geq M(u, v, w, t).
\]

Similarly (1) we have

\[
M(Au, y_{2n+1}, kt) = M(Au, Bx_{2n-1}, Cx_{2n}, kt)
\]

Taking the limit as \( n \to \infty \),

\[
M(Au, z, z, kt) = M(Au, z, z, kt) \geq \left\{ \begin{array}{l}
M(Ru, z, z, t) \cdot M(Au, z, z, t) \cdot M(z, z, Au, t) \\
M(z, z, z, t) \cdot M(z, z, Au, t) \\
M(Au, z, z, t) \cdot M(z, z, z, t) \\
M(z, Au, z, t) \cdot M(z, z, z, t) \cdot 1 \cdot M(Au, z, z, t) \cdot 1 \cdot 1 \\
\end{array} \right.
\]

which gives \( M(Au, z, z, kt) \geq M(Au, z, z, t) \)

Therefore by lemma (2.7), we have \( Au = z \). Since \( Ru = z \).

Thus \( Au = Ru = z \), that is, \( u \) is a coincidence point of \( R \) and \( A \).

Similarly (1) we have

\[
M(y_{2n-1}, Bu, y_{2n}, kt) = M(Ax_{2n-2}, Bu, Px_{2n}, kt)
\]

Taking the limit as \( n \to \infty \),

\[
M(z, Bu, z, kt) = M(z, Bu, z, kt) \geq \left\{ \begin{array}{l}
M(z, Qv, z, t) \cdot M(z, z, z, t) \cdot M(Bu, Qv, z, t) \\
M(z, z, z, t) \cdot M(z, Qv, z, t) \cdot M(Bu, z, z, t) \\
M(z, z, Bu, t) \cdot M(z, z, Bu, t) \cdot M(Bu, z, z, t) \\
M(z, z, z, t) \cdot M(Bu, z, z, t) \cdot M(z, z, z, t) \\
M(z, Qv, z, t) \cdot 1 \cdot M(Bu, Qv, z, t) \cdot 1 \cdot M(z, Qv, z, t) \\
M(Bu, z, z, t) \cdot M(z, z, Bu, t) \cdot M(z, z, Bu, t) \\
M(Bu, z, z, t) \cdot 1 \cdot M(Bu, z, z, t) \cdot M(z, z, z, t) \\
\end{array} \right.
\]

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Which gives \( M(z, Bv, z, kt) \geq M(Bv, z, z, t) \) therefore by lemma(2.7). We have \( Bv = z \).
Since \( Qv = z \). Thus \( Bv = Qv = z \), that is, \( v \) is a coincidence point of \( B \) and \( v \).
Similarly (1), we have
\[
M(y_{2n+1}, y_{2n}, Cw, kt) = M(Cx_{2n}, Bx_{2n-1}, Cw, kt)
\]
\[
\begin{align*}
&\geq \left\{ \\
&M(Rx_{2n+1}, Qx_{2n-1}, Pw, t) * M(Ax_{2n-2}, Rx_{2n-1}, Cw, t) * M(Bx_{2n-1}, Qx_{2n-1}, Ax_{2n-2}, t) \\
&* M(Cw, Pw, Bx_{2n-1}, t) * M(Ax_{2n-2}, Qx_{2n-1}, Cw, t) * M(Bx_{2n-1}, Pw, Ax_{2n-2}, t) \\
&* M(Ax_{2n-2}, Qx_{2n-1}, Bx_{2n-1}, t) * M(Bx_{2n-1}, Pw, Cx_{2n}, t) * M(Cw, Pw, Ax_{2n-2}, t)
\end{align*}
\]

Taking the limit as \( n \to \infty \),
\[
M(z, z, Cw, kt) = M(z, z, Cw, kt) \geq \left\{ \\
&M(z, z, Pw, t) * M(z, z, Cw, t) * M(z, z, z, t) \\
&* M(Cw, Pw, z, t) * M(z, z, Cw, t) * M(z, Pw, z, t) \\
&* M(Cw, z, z, t) * M(z, z, z, t) * M(z, z, z, t) \\
&* M(z, z, z, t) * M(z, z, z, t) * M(Cw, Pw, z, t)
\right\}
\]

Which gives \( M(z, z, Cw, kt) \geq M(Cw, z, z, t) \) therefore by lemma(2.7). We have \( Cw = z \).
Since \( Pw = z \). Thus \( Cw = Rw = z \), that is, \( w \) is a coincidence point of \( C \) and \( w \).
Since the pair \( \{ A, R \} \) is weakly compatible, therefore \( A \) and \( R \) commute at their coincidence point that is \( A(Ru) = R(Au) \) or \( Az = Rz \).
Similarly the pair \( \{ B, Q \} \) is weakly compatible, therefore \( B \) and \( Q \) commute at their coincidence point that is \( B(Qw) = Q(Bw) \) or \( Bz = Qz \).
Similarly the pair \( \{ C, P \} \) is weakly compatible, therefore \( C \) and \( P \) commute at their coincidence point that is \( C(Pw) = P(Cw) \) or \( Cz = Pz \).
Now we prove \( Az = z \). By (1), we have
\[
M(Az, Bx_{2n-1}, Cx_{2n}, kt)
\]
\[
\begin{align*}
&\geq \left\{ \\
&M(Rx_{2n+1}, Qx_{2n-1}, Pw, t) * M(Az, Rx_{2n+1}, Cx_{2n}, t) * M(Bx_{2n-1}, Qx_{2n-1}, Az, t) \\
&* M(Cx_{2n}, Pw, Bx_{2n-1}, t) * M(Ax_{2n-2}, Qx_{2n-1}, Cx_{2n}, t) * M(Bx_{2n-1}, Pw, Ax_{2n-2}, t) \\
&* M(Az, Qx_{2n-1}, Bx_{2n-1}, t) * M(Bx_{2n-1}, Pw, Cx_{2n}, t) * M(Cx_{2n}, Pw, Az, t)
\end{align*}
\]

Taking the limit as \( n \to \infty \),
\[
M(Az, z, z, kt) \geq \left\{ \\
&M(z, z, z, t) * M(Az, z, z, t) * M(z, z, Az, t) \\
&* M(z, z, z, t) * M(z, z, z, t) * M(z, z, z, t) \\
&* M(z, z, z, t) * M(z, z, z, t) * M(z, z, z, t) \\
&* M(Az, z, z, t) * M(z, z, Az, t)
\right\}
\]
\[
M(Az, z, z, kt) \geq \left\{ \\
1 * M(Az, z, z, t) * M(z, z, Az, t) * 1 * 1 * M(z, z, z, t) \\
* M(Az, z, z, t) * 1 * M(Az, z, z, t) * 1 * M(z, z, Az, t)
\right\}
\]
\[
M(Az, z, z, kt) \geq \left\{ \\
1 * M(Az, z, z, t) * M(z, z, Az, t) * 1 * 1 * 1 * 1 * 1 * 1 *
\right\}
\]

\[
M(Az, z, z, kt) \geq \left\{ \\
1 * M(Az, z, z, t) * M(z, z, Az, t) * 1 * 1 * M(z, z, Az, t)
\right\}
\]

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Which gives \( M(Az, z, z, kt) \geq M(Az, z, z, t) \) therefore by lemma (2.7). We have \( Az = z \).
Since \( Rz = Az \). Thus \( Az = Rz = z \). Now we prove \( Bz = z \).
Similarly (1), we have
\[
M(Ax_{2n-2}, Bz, Cx_{2n}, kt)
\]
\[
\geq \begin{cases} 
M(Rx_{2n+1}, Qz, Px_{2n}, t) * M(Ax_{2n-2}, Rx_{2n-1}, Cx_{2n}, t) * M(Bz, Qz, Ax_{2n-2}, t) \\
* M(Cx_{2n}, Px_{2n}, Bz, t) * M(Ax_{2n}, Qz, Cx_{2n}, t) * M(Bz, Px_{2n}, Ax_{2n-2}, t) \\
* M(Cx_{2n}, Rx_{2n+1}, Bz, t) * M(Ax_{2n-2}, Px_{2n}, Bz, t) * M(Bz_{2n-1}, Rx_{2n+1}, Cx_{2n}, t) \\
* M(Ax_{2n-2}, Qx_{2n-1}, Bx_{2n-1}, t) * M(Bz, Px_{2n}, Cx_{2n}, t) * M(Cx_{2n}, Px_{2n}, Ax_{2n-2}, t)
\end{cases}
\]

Taking the limit as \( n \to \infty \),
\[
M(z, Bz, z, kt) = M(z, Bz, z, kt)
\geq \begin{cases} 
M(z, z, z, t) * M(z, z, z, t) * M(Bz, Qz, z, t) \\
* M(z, z, Bz, t) * M(z, Qz, z, t) * M(Bz, z, t) \\
* M(z, z, Bz, t) * M(z, z, Bz, t) * M(Bz, z, t) \\
* M(z, z, z, t) * M(Bz, z, z, t) \\
* M(z, z, z, t) * M(z, z, z, t) * M(Bz, z, z, t) \\
* M(z, z, z, t) * M(Bz, z, z, t)
\end{cases}
\]

Which gives \( M(z, Bz, z, kt) \geq M(Bz, z, z, t) \) therefore by lemma (2.7). We have \( Bz = z \).
Since \( Qz = Bz \). Thus \( Bz = Qz = z \).
Now we prove \( Cz = z \).
Similarly (1) we have
\[
M(Ax_{2n-2}, Bx_{2n-1}, Cz, kt)
\]
\[
\geq \begin{cases} 
M(Rx_{2n+1}, Qx_{2n-1}, Pz, t) * M(Ax_{2n-2}, Rx_{2n-1}, Cz, t) * M(Bz_{2n-1}, Qx_{2n-1}, Ax_{2n-2}, t) \\
* M(Cz, Pz, Bx_{2n-1}, t) * M(Ax_{2n-2}, Qx_{2n-1}, Cz, t) * M(Bz_{2n-1}, Pz, Ax_{2n-2}, t) \\
* M(Cz, Rx_{2n+1}, Bx_{2n-1}, t) * M(Ax_{2n-2}, Px_{2n}, Bx_{2n-1}, t) * M(Bz_{2n-1}, Rx_{2n+1}, Cx_{2n}, t) \\
* M(Ax_{2n-2}, Qx_{2n-1}, Bx_{2n-1}, t) * M(Bz_{2n-1}, Px_{2n}, Cx_{2n}, t) * M(Cz, Pz, Ax_{2n-2}, t)
\end{cases}
\]

Taking the limit as \( n \to \infty \),
\[
M(z, z, Cz, kt) \geq \begin{cases} 
M(z, z, Pz, t) * M(z, z, Cz, t) * M(z, z, z, t) \\
* M(Cz, Pz, z, t) * M(z, z, Cz, t) * M(z, Pz, z, t) \\
* M(Cz, z, z, t) * M(z, z, z, t) * M(z, z, z, t) \\
* M(z, z, z, t) * M(z, z, z, t) * M(Cz, Pz, z, t)
\end{cases}
\]
\[ M(z, z, Cz, kt) \geq \begin{cases} 
M(z, z, Cz, t) \ast M(z, z, Cz, t) \ast M(z, z, z, t) \\
\ast M(Cz, z, z, t) \ast M(z, z, z, t) \\
\ast M(z, z, z, t) \ast M(Cz, z, z, t) \\
\end{cases} \]

\[ M(z, z, Cz, kt) \geq \begin{cases} 
1 \ast M(z, z, Cz, t) \ast M(Cz, z, z, t) \ast M(z, z, z, t) \\
\ast 1 \ast M(Cz, z, z, t) \ast M(Cz, z, z, t) \\
\end{cases} \]

Which gives \( M(z, z, Cz, kt) \geq M(Cz, z, z, t) \)

therefore by lemma(2.7). We have \( Cz = z \). Since \( Pz = Cz \) Thus \( Cz = Pz = z \).

Uniqueness:
Let \( w' \) be another common fixed point of \( A, B, C, P, Q \) and \( R \). By (1) we have

\[ M(z, w', z, kt) = M(Az, Bw', Cz, kt) \]

\[ \geq \begin{cases} 
M(Rw', Qz, Pz, t) \ast M(Az, Rw', Cz, t) \ast M(Bw', Qz, Az, t) \\
\ast M(Cz, Pz, Bz, t) \ast M(Aw', Qw', Cz, t) \ast M(Bw', Pz, Az, t) \\
\ast M(Cz, Rz, Bw', t) \ast M(Az, Pz, Bw', t) \ast M(Bw', Rz, Cz, t) \\
\ast M(Az, Qz, Bz, t) \ast M(Bw', Pz, Cz, t) \ast M(Cz, Pz, Az, t) \\
\end{cases} \]

\[ M(z, w', z, kt) = M(z, Bw', z, kt) \geq \begin{cases} 
M(Rw', z, z, t) \ast M(z, Rw', z, t) \ast M(Bw', z, z, t) \\
\ast M(z, z, z, t) \ast M(Aw', Qu', z, t) \ast M(Bw', z, z, t) \\
\ast M(z, z, Bw', t) \ast M(z, z, Bw', t) \ast M(Bw', z, z, t) \\
\ast M(z, z, z, t) \ast M(Bw', Pz, Cz, t) \ast M(z, z, z, t) \\
\end{cases} \]

\[ M(z, w', z, kt) = M(z, w', z, kt) \geq \begin{cases} 
M(w', z, z, t) \ast M(z, w', z, t) \ast M(w', z, z, t) \\
\ast M(z, z, z, t) \ast M(w', w', z, t) \ast M(w', z, z, t) \\
\ast M(z, z, w', t) \ast M(z, z, w', t) \ast M(w', z, z, t) \\
\ast M(z, z, z, t) \ast M(w', z, z, t) \ast M(z, z, z, t) \\
\end{cases} \]

From lemma (2.7) \( z = w' \)

Therefore \( z \) is common fixed point of \( A, B, C, P, Q, \) and \( R \).

References


