FUZZY INTERVAL INTEGER TRANSPORTATION PROBLEMS

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Abstract
A new method namely, level-bound method for solving fully fuzzy interval integer transportation problems is proposed. Numerical example is shown for understanding the solution procedure of the proposed method.

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Key Words: Fuzzy sets, Fuzzy interval number, Transportation problem, Optimal solution, Level-bound method.

1 Introduction
Transportation problem (TP) is one of the popular and most important applications of the linear programming problem. Many efficient algorithms have been developed for solving TPs having deterministic parameters. In many real life situations, some or all parameters of the TP are not deterministic always, but they are uncertain. Zimmermann [19] developed Zimmermann's fuzzy linear programming into several fuzzy optimization methods for solving the TPs. Chanas et al. [2] formulated the classical, interval and fuzzy transportation problem and discussed the methods for solution for the fuzzy TP. Many researchers [4, 5, 8, 11, 13, 17] have proposed various methods to solve interval and fuzzy TPs. Pawlak [15] initiated the rough set theory. Then, many researchers have developed the rough set theory both in theoretical and applied. A rough programming problem considering the decision set
as a rough set was introduced and solved by Youness [18]. Some solid transportation models with crisp and rough costs were solved by Kundu et al. [9]. Subhakanta Dash and Mohanty [16] have proposed a compromise solution method for transportation problems considering the unit cost of transportation from a source to a destination as a rough integer interval. Various methods for solving interval integer transportation problems with rough nature are presented in Akilbasha et al. [1] and Pandian et al. [14].

In this paper, we present a new method namely, level-bound method to find an optimal solution for integer TP s where transportation cost, supply and demand are fuzzy integer intervals. The proposed method is based on the transportation algorithm [6]/the integer linear programming technique [6]. The optimal objective value and the optimal decision values of variables for the fuzzy interval integer TP given by the level-bound method are fuzzy interval integers. The solution procedure is demonstrated using numerical examples. The level-bound method helps the decision makers for solving transportation models of real life situations in which all or some parameters are fuzzy interval integers.

2 Preliminaries

We have used the definitions, basic arithmetic operators and partial ordering relate to real intervals and fuzzy sets which can be found in [6, 10] this paper.

The following definitions of membership function, the basic arithmetic operators and partial ordering on closed bounded fuzzy intervals [5] based on the definitions in real interval sets and fuzzy sets are defined.

Let $\tilde{D}$ denote the set of all closed bounded fuzzy intervals over $F(R)$.

That is, $\tilde{D} = \{ [\tilde{a}, \tilde{b}], \tilde{a} = (a_1, a_2, a_3) \text{ and } \tilde{b} = (b_1, b_2, b_3), a_3 \leq b_1 \text{ and } a_i \text{’}s \text{ and } b_i \text{’}s \text{ are in } R \}.

The membership function $\mu_{[\tilde{a}, \tilde{b}]}(x)$ of the fuzzy interval number
\[ [\tilde{a}, \tilde{b}] \text{ is given below.} \]

\[
\mu_{[\tilde{a}, \tilde{b}]}(x) = \begin{cases} 
(x - a_1)/(a_2 - a_1) : a_1 \leq x \leq a_2 \\
(a_3 - x)/(a_3 - a_2) : a_2 \leq x \leq a_3 \\
(x - b_1)/(b_2 - b_1) : b_1 \leq x \leq b_2 \\
(b_3 - x)/(b_3 - b_2) : b_2 \leq x \leq b_3 \\
0 : \text{otherwise}
\end{cases}
\]

The graph of \( \mu_{[\tilde{a}, \tilde{b}]}(x) \) is given below:

\[
\begin{array}{c}
\hline
& \tilde{a} & a_1 & a_2 & b_1 & b_2 & b_3 \\
\hline
0 & & & & & & \\
0.5 & & & & & & \\
1 & & & & & & \\
\hline
\end{array}
\]

**Definition 1.** Let \( \tilde{A} = [\tilde{a}, \tilde{b}] \) and \( \tilde{B} = [\tilde{c}, \tilde{d}] \) be in \( \tilde{D} \). Then,
\( (i) \tilde{A} \oplus \tilde{B} = [a + c, b + d] \); \( (ii) \tilde{A} \Theta \tilde{B} = [a - d, b - c] \); \( (iii) k\tilde{A} = [k\tilde{a}, k\tilde{b}] \)
if \( k \) is a positive real number; \( (iv) k\tilde{A} = [k\tilde{b}, k\tilde{a}] \) if \( k \) is a negative real number and \( (v) \tilde{A} \otimes \tilde{B} = [\tilde{p}, \tilde{q}] \) where \( \tilde{p} = (p_1, p_2, p_3), \tilde{q} = (q_1, q_2, q_3) \),
\( p_1 = \min \{a_1c_1, a_1d_1, b_1c_1, b_1d_1\} \), \( q_1 = \max \{a_1c_1, a_1d_1, b_1c_1, b_1d_1\} \),
\( p_2 = \min \{a_2c_2, a_2d_2, b_2c_2, b_2d_2\} \), \( q_2 = \max \{a_2c_2, a_2d_2, b_2c_2, b_2d_2\} \),
\( p_3 = \min \{a_3c_3, a_3d_3, b_3c_3, b_3d_3\} \) and \( q_3 = \max \{a_3c_3, a_3d_3, b_3c_3, b_3d_3\} \).

**Definition 2.** Let \( \tilde{A} = [\tilde{a}, \tilde{b}] \) and \( \tilde{B} = [\tilde{c}, \tilde{d}] \) be in \( \tilde{D} \). Then,
\( (i) \tilde{A} \leq \tilde{B} \) if and only if \( \tilde{a} \leq \tilde{c} \) and \( \tilde{b} \leq \tilde{d} \),
\( (ii) \tilde{A} = \tilde{B} \) if and only if \( \tilde{a} = \tilde{c} \) and \( \tilde{b} = \tilde{d} \).

**Definition 3.** Let \( \tilde{A} = [\tilde{a}, \tilde{b}] \) be in \( \tilde{D} \). Then, \( (i) \tilde{A} \) is said to be positive if \( \tilde{a} \geq 0 \); \( (ii) \tilde{A} \) is said to be integer if \( \tilde{a} \) and \( \tilde{b} \) are integers.

### 3 Fully Fuzzy Interval Integer TP

Consider the following fuzzy interval integer TP problem:

(P) Minimize \( \tilde{Z}, \tilde{W} = \sum_{i=1}^{m} \sum_{j=1}^{n} [c_{ij}, \tilde{d}_{ij}] \otimes [\tilde{x}_{ij}, \tilde{y}_{ij}] \)

Subject to
\[
\sum_{j=1}^{n} \tilde{x}_{ij} = [\tilde{a}_i, \tilde{p}_i], \text{ for } i \in I \\
(1)
\]
\[\sum_{i=1}^{m} [\tilde{x}_{ij}, \tilde{y}_{ij}] = [\tilde{b}_{j}, \tilde{q}_{j}], \text{ for } j \in J \] (2)
\[\tilde{x}_{ij}, \tilde{y}_{ij} \succ 0 \text{ and integers, for } i \in I \text{ and } j \in J \] (3)
where \(I = \{1, 2, \ldots, m\}, J = \{1, 2, \ldots, n\}\), \(m = \) the number of supply points; \(n = \) the number of demand points; \([\tilde{x}_{ij}, \tilde{y}_{ij}]\) is the fuzzy interval number of units shipped from supply point \(i\) to demand point \(j\) with \(\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3)\) and \(\tilde{y}_{ij} = (y_{ij}^1, y_{ij}^2, y_{ij}^3)\);
\([\tilde{c}_{ij}, \tilde{d}_{ij}]\) is the fuzzy interval cost of shipping one unit from supply point \(i\) to the demand point \(j\) with \(\tilde{c}_{ij} = (c_{ij}^1, c_{ij}^2, c_{ij}^3)\) and \(\tilde{d}_{ij} = (d_{ij}^1, d_{ij}^2, d_{ij}^3)\);
\([\tilde{a}_i, \tilde{p}_i]\) is the fuzzy interval supply at supply point \(i\) with \(\tilde{a}_i = (a_i^1, a_i^2, a_i^3)\) and \(\tilde{p}_i = (p_i^1, p_i^2, p_i^3)\); and \([\tilde{b}_j, \tilde{q}_j]\) is the fuzzy interval demand at demand point \(j\) with \(\tilde{b}_j = (b_j^1, b_j^2, b_j^3)\) and \(\tilde{q}_j = (q_j^1, q_j^2, q_j^3)\).

In the problem (P), if the total supply is equal to the total demand, the problem (P) is said to be balanced.

**Definition 4.** A set of fuzzy intervals \([[(x^1_{ij}, x^2_{ij}, x^3_{ij}), (y^1_{ij}, y^2_{ij}, y^3_{ij})]]\), for all \(i \in I\) and \(j \in J\) is said to be a feasible solution to the problem (P) if it satisfies the equations (1), (2) and (3).

**Definition 5.** A feasible solution \([[(x^1_{ij}, x^2_{ij}, x^3_{ij}), (y^1_{ij}, y^2_{ij}, y^3_{ij})]]\), for all \(i \in I\) and \(j \in J\) to the problem (P) is said to be an optimal solution of the problem (P) if the feasible solution minimizes the objective function of the problem (P), that is,
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} [([c^1_{ij}, c^2_{ij}, c^3_{ij}), ([d^1_{ij}, d^2_{ij}, d^3_{ij}])] \otimes ([x^1_{ij}, x^2_{ij}, x^3_{ij}], [y^1_{ij}, y^2_{ij}, y^3_{ij}])
\[
\leq \sum_{i=1}^{m} \sum_{j=1}^{n} [([c^1_{ij}, c^2_{ij}, c^3_{ij}), ([d^1_{ij}, d^2_{ij}, d^3_{ij}])] \otimes ([u^1_{ij}, u^2_{ij}, u^3_{ij}], [v^1_{ij}, v^2_{ij}, v^3_{ij}])
\text{ for all feasible } \{([u^1_{ij}, u^2_{ij}, u^3_{ij}], [v^1_{ij}, v^2_{ij}, v^3_{ij}])\}, \text{ for all } i \in I \text{ and } j \in J \}
\text{ to the problem (P).}
\]

Now, the problem (P) is levelled into six integer TPs namely, 6th level integer TP (TP6), 5th level integer TP (TP5), 4th level integer TP (TP4), 3rd level integer TP (TP3), 2nd level integer TP (TP2) and 1st level integer TP (TP1) as given below:

(TP6) Minimize \(w_3 = \sum_{i=1}^{m} \sum_{j=1}^{n} d^3_{ij} y^3_{ij}\)

Subject to \(\sum_{j=1}^{n} y^3_{ij} = p^i_1, i \in I; \sum_{i=1}^{m} y^3_{ij} = q^j_1, j \in J; y^3_{ij} \geq 0\)

\[4\]

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Minimize $w_2 = \sum_{i=1}^m \sum_{j=1}^n d_{ij}y_{ij}^2$

Subject to $\sum_{j=1}^n y_{ij}^2 = p_i^2$, $i \in I$; $\sum_{i=1}^m y_{ij}^2 = q_j^2$, $j \in J$; $y_{ij}^2 \geq 0$

Minimize $w_1 = \sum_{i=1}^m \sum_{j=1}^n d_{ij}y_{ij}^1$

Subject to $\sum_{j=1}^n y_{ij}^1 = p_i^1$, $i \in I$; $\sum_{i=1}^m y_{ij}^1 = q_j^1$, $j \in J$; $y_{ij}^1 \geq 0$

Minimize $z_3 = \sum_{i=1}^m \sum_{j=1}^n c_{ij}^3 x_{ij}^3$

Subject to $\sum_{j=1}^n x_{ij}^3 = a_i^3$, $i \in I$; $\sum_{i=1}^m x_{ij}^3 = b_j^3$, $j \in J$; $x_{ij}^3 \geq 0$

Minimize $z_2 = \sum_{i=1}^m \sum_{j=1}^n c_{ij}^2 x_{ij}^2$

Subject to $\sum_{j=1}^n x_{ij}^2 = a_i^2$, $i \in I$; $\sum_{i=1}^m x_{ij}^2 = b_j^2$, $j \in J$; $x_{ij}^2 \geq 0$, and

Minimize $z_1 = \sum_{i=1}^m \sum_{j=1}^n c_{ij}^1 x_{ij}^1$

Subject to $\sum_{j=1}^n x_{ij}^1 = a_i^1$, $i \in I$; $\sum_{i=1}^m x_{ij}^1 = b_j^1$, $j \in J$; $x_{ij}^1 \geq 0$,

for all $i \in I$ and $j \in J$ and are integers.

Now, we establish a relation between optimal solutions of the fully fuzzy interval integer TP(P) and its six induced integer TPs, (TP6) to (TP1). The established relation is employed in the proposed method, namely, level-bound method.

**Theorem 6.** If the set $\{\bar{y}_{ij}^3, \bar{y}_{ij}^1, \bar{y}_{ij}^2\}$, for all $i \in I$ and $j \in J$ is an optimal solution for the problem (TP6) with the minimum transportation cost $\bar{w}_3$, the set $\{\bar{y}_{ij}^3, \bar{y}_{ij}^1, \bar{y}_{ij}^2\}$, for all $i \in I$ and $j \in J$ is an optimal solution for the problem (TP5) with the minimum transportation cost $\bar{w}_2$, the set $\{\bar{y}_{ij}^3, \bar{y}_{ij}^1, \bar{y}_{ij}^2\}$, for all $i \in I$ and $j \in J$ is an optimal solution for the problem (TP4) with the minimum transportation cost $\bar{w}_1$, the set $\{\bar{x}_{ij}^3, \bar{x}_{ij}^1, \bar{x}_{ij}^2\}$, for all $i \in I$ and $j \in J$ is an optimal solution for the problem (TP3) with the minimum transportation cost $\bar{z}_3$, the set $\{\bar{x}_{ij}^3, \bar{x}_{ij}^1, \bar{x}_{ij}^2\}$, for all $i \in I$ and $j \in J$ is an optimal solution for the problem (TP2) with the minimum transportation cost $\bar{z}_2$ and the set $\{\bar{x}_{ij}^3, \bar{x}_{ij}^1, \bar{x}_{ij}^2\}$, for all $i \in I$ and $j \in J$ is an optimal solution for the problem (TP1) with the minimum transportation cost $\bar{z}_1$, then the set of fuzzy integer intervals $\{[(\bar{x}_{ij}^1, \bar{x}_{ij}^2, \bar{x}_{ij}^3), (\bar{y}_{ij}^1, \bar{y}_{ij}^2, \bar{y}_{ij}^3)]\}$, for all
\[ i \in I \text{ and } j \in J \} \text{ is an optimal solution for the problem } (P) \text{ with the minimum transportation cost } \{(\bar{z}_1, \bar{z}_2, \bar{z}_3), (\bar{w}_1, \bar{w}_2, \bar{w}_3)\} \text{ provided } \bar{x}_{ij}^1 \leq \bar{x}_{ij}^2 \leq \bar{x}_{ij}^3 \leq \bar{y}_{ij}^1 \leq \bar{y}_{ij}^2 \leq \bar{y}_{ij}^3, \text{ for all } i \in I \text{ and } j \in J. \]

Proof. It is proved easily from basic definitions of fuzzy interval numbers and feasible and optimal conditions of the fuzzy interval transportation problem. \qed

4 Level-Bound Method

We, now propose a new method namely, level-bound method for solving the fully fuzzy interval integer TP(P).

The level-bound method proceeds as follows.

Step 1: Check that the given problem (P) is balanced. If not, make it into balanced.

Step 2: Construct six level TP problems of the given problem (P).

Step 3: Solve the 6th level TP problem using a transportation algorithm [7, 12]. Let \{\bar{y}_{ij}^3, \text{ for all } i \in I \text{ and } j \in J\} be an optimal solution of the 6th level TP problem with the minimum transportation cost \(\bar{w}_3\).

Step 4: Solve the 5th level TP problem with the upper bound constraints \(y_{ij}^2 \leq \bar{y}_{ij}^3\), for all \(i \in I \text{ and } j \in J\) using the zero point method [12] / the integer linear programming technique [7]. Let \{\bar{y}_{ij}^2, \text{ for all } i \in I \text{ and } j \in J\} be an optimal solution of the 5th level TP problem with the minimum transportation cost \(\bar{w}_2\).

Step 5: Solve the 4th level TP problem with the upper bound constraints \(y_{ij}^1 \leq \bar{y}_{ij}^3\), for all \(i \in I \text{ and } j \in J\) using the zero point method [12] / the integer linear programming technique [7]. Let \{\bar{y}_{ij}^1, \text{ for all } i \in I \text{ and } j \in J\} be an optimal solution of the 4th level TP problem with the minimum transportation cost \(\bar{w}_1\).

Step 6: Solve the 3rd level TP problem with the upper bound constraints \(x_{ij}^3 \leq \bar{y}_{ij}^3\), for all \(i \in I \text{ and } j \in J\) using the zero point method [12] / the integer linear programming technique [7]. Let \{\bar{x}_{ij}^3, \text{ for all } i \in I \text{ and } j \in J\} be an optimal solution of the 3rd level TP problem with the minimum transportation cost \(\bar{z}_3\).

Step 7: Solve the 2nd level TP problem with the upper bound constraints \(x_{ij}^2 \leq \bar{x}_{ij}^3\), for all \(i \in I \text{ and } j \in J\) using the zero point method [12] / the integer linear programming technique [7]. Let \{\bar{x}_{ij}^2, \text{ for all } i \in I \text{ and } j \in J\} be an optimal solution of the 2nd level
TP problem with the minimum transportation cost $\bar{z}_2$.

**Step 8**: Solve the 1st level TP problem with the upper bound constraints $x_{ij}^1 \leq \bar{x}_{ij}$, for all $i \in I$ and $j \in J$ using the zero point method [12] / the integer linear programming technique [7]. Let \{ $x_{ij}^1$ \} for all $i \in I$ and $j \in J$ be an optimal solution of the 1st level TP problem with the minimum transportation cost $\bar{z}_1$.

**Step 9**: The optimal solution of the given problem (P) is \{ ($x_{ij}^2, \bar{x}_{ij}^2, \bar{x}_{ij}^3$, ($\bar{y}_{ij}^2, \bar{y}_{ij}^2, \bar{y}_{ij}^3$)) \}, for all $i \in I$ and $j \in J$ with the minimum transportation cost \{ ($\bar{z}_1, \bar{z}_2, \bar{z}_3$, ($\bar{w}_1, \bar{w}_2, \bar{w}_3$)) \} (by Theorem 6).

The solution procedure of the proposed method for solving the fully fuzzy interval integer TP is illustrated by the following numerical example.

**Example 1**: Consider the following fully fuzzy interval integer TP:

<table>
<thead>
<tr>
<th></th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>[5, 6, 7],</td>
<td>[10, 11, 12],</td>
<td>[7, 8, 9],</td>
<td>[11, 12, 13],</td>
</tr>
<tr>
<td></td>
<td>(9, 10, 11)</td>
<td>(14, 15, 16)</td>
<td>(11, 12, 13)</td>
<td>(15, 16, 17)</td>
</tr>
<tr>
<td>S2</td>
<td>[1, 2, 4],</td>
<td>[2, 3, 5],</td>
<td>[4, 6, 7],</td>
<td>[9, 10, 11],</td>
</tr>
<tr>
<td></td>
<td>(5, 6, 7)</td>
<td>(6, 7, 8)</td>
<td>(8, 9, 10)</td>
<td>(13, 14, 15)</td>
</tr>
<tr>
<td>S3</td>
<td>[0, 3, 4],</td>
<td>[0, 1, 2],</td>
<td>[8, 9, 10],</td>
<td>[12, 13, 14],</td>
</tr>
<tr>
<td></td>
<td>(6, 7, 8)</td>
<td>(3, 4, 5)</td>
<td>(11, 12, 13)</td>
<td>(16, 18, 19)</td>
</tr>
<tr>
<td>Demand</td>
<td>[18, 19, 20],</td>
<td>[9, 10, 11],</td>
<td>[5, 6, 7],</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(22, 24, 25)</td>
<td>(13, 14, 15)</td>
<td>(9, 10, 11)</td>
<td></td>
</tr>
</tbody>
</table>

Now, since the total supply = the total demand = [(32, 35, 38), (44, 48, 51)], the given problem is balanced.

Now, by using the proposed level-bound method we can get the optimal solution of the problem (P) is given below

\[ (\bar{x}_{11}^1, \bar{x}_{12}^1, \bar{x}_{13}^1, \bar{y}_{11}^2, \bar{y}_{12}^2, \bar{y}_{13}^2) = (6, 6, 6, 6, 6, 6), \]
\[ (\bar{x}_{12}^1, \bar{x}_{12}^2, \bar{x}_{13}^2, \bar{y}_{12}^2, \bar{y}_{13}^2) = (5, 6, 7, 9, 10, 11), \]
\[ (\bar{x}_{21}^1, \bar{x}_{21}^2, \bar{x}_{21}^3, \bar{y}_{21}^2, \bar{y}_{21}^3) = (9, 10, 11, 13, 14, 15), \]
\[ (\bar{x}_{21}^1, \bar{x}_{21}^2, \bar{x}_{22}^2, \bar{y}_{21}^2, \bar{y}_{22}^2) = (3, 3, 3, 3, 4), \]
\[ (\bar{x}_{22}^1, \bar{x}_{22}^2, \bar{x}_{22}^3, \bar{y}_{22}^2, \bar{y}_{22}^3) = (9, 10, 11, 13, 14, 15), \]

with the minimum shipping cost [(74, 123, 186), (275, 348, 421)].

## 5 Conclusion

In this paper, a new method namely, level-bound method is proposed to solve fuzzy interval integers TP in which all of the parameters are fuzzy interval integers has been presented. The level-bound
The method is a systematic procedure, both easy to understand and to apply and also, it is a crisp method and provides an exact optimal solution to the given problem. Numerical example is presented to understand the proposed method. The developed method can be served an important tool for the decision makers when they are handling various types of logistic problems having fuzzy interval integer parameters.

References


