DOMINATOR CHROMATIC NUMBER OF SOME GRAPHS

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ABSTRACT

The dominator coloring of a graph G is defined as a proper coloring of G where each vertex of the graph dominates every vertex of at least one color class. The dominator chromatic number is the least number of color classes used in a dominator coloring of a graph G. The dominator chromatic number and domination number of dragon graph and lollipop graph are obtained and a relationship between the chromatic number, domination number and dominator chromatic number is expressed in this paper.

Keywords: Coloring, Domination, Dominator coloring, Dragon graph, Lollipop graph

1. INTRODUCTION

Let G be a graph with vertex set V and edge set E. A dominating set S is a subset of the vertex set V such that every vertex in the graph either belongs to S or has a neighbor in S. If no proper subset of S is a dominating set, then S is called a minimum dominating set. The domination number denoted by γ(G) is the order of a minimum dominating set. Dominating sets are useful in routing problems, scheduling problems, assignment problems, computer communication networks, land surveying, assignment problems, coding theory etc..

A proper vertex coloring of a graph G is allocation of colors to the vertices such that two neighboring vertices belong to the different color class. The chromatic number, denoted by χ(G) is the minimum number of colors required among all proper colorings of G. Graph coloring are useful tools in modeling many practical problems such as scheduling, allocation of resources, assignment problems, pattern matching, circuit testing etc..

Graph coloring and domination have numerous applications to today’s networks. The area obtained by combining graph coloring and domination is called dominator coloring. A dominator coloring of graph G is an assignment of colors to the vertices of G such that two neighboring vertices are assigned different colors and every vertex dominates all vertices of at least one color class. The minimum

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number of color classes required for a dominator coloring of $G$ is called dominator chromatic number, denoted by $\chi_d(G)$.

The concept of dominator coloring was introduced in 2006 [1]. The relation between dominator coloring, proper coloring and domination number of different classes of graphs were shown in [2], [3], [5]. The dominator coloring of prism graph, quadrilateral snake, triangle snake and barbell graph, proper interval graphs, block graphs were also studied in various papers [6], [7], [8]. The algorithmic aspects of dominator colorings in graphs have been discussed by Arumugam S etal. in [4].

A $(m, n)$ dragon graph denoted by $D(m, n)$ is the graph obtained by combining a cycle graph $C_m$ to a path graph $P_n$ with a bridge. It has $(m + n)$ vertices and $(m + n)$ edges. It is also called as a tadpole graph. The $(m, 1)$-dragon graph is sometimes known as the m-pan graph. The particular cases of $(3, 1)$ and $(4, 1)$ dragon graphs are known as the paw graph and banner graph respectively.

A $(m, n)$ lollipop graph denoted by $L(m, n)$ is obtained by connecting a complete graph $K_m$ to a path graph $P_n$ with a bridge. It has $(m + n)$ vertices and \[ \left\lceil \frac{m(m-1)}{2} \right\rceil + n \] edges.

In this paper the domination number and dominator chromatic number of dragon graph and lollipop graph are obtained and a relation between the dominator chromatic number, domination number and chromatic number is expressed.

## 2. DOMINATOR CHROMATIC NUMBER OF DRAGON GRAPH

**Theorem 2.1:** The dominator chromatic number of dragon graph $D(m, n)$ when $m \geq 3$ and $\forall n$ is given by

\[
\chi_d(D(m, n)) = \begin{cases} 
3 + \left\lceil \frac{n-1}{2} \right\rceil & \text{if } m = 4, 1 \leq n \leq 5 \\
4 + \left\lceil \frac{n-1}{3} \right\rceil & \text{if } m = 4, n \geq 6 \\
\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil + 2 & \text{otherwise}
\end{cases}
\]

\[(1)\]

**Proof:**

Consider a dragon graph $D(m, n)$ obtained by combining a cycle graph $C_m$ to a path graph $P_n$ with a bridge. It has $(m + n)$ vertices and $(m + n)$ edges. The vertex set $V$ and the edge set $E$ are given by

\[ V = \{v_i/1 \leq i \leq m\} \cup \{u_j/1 \leq j \leq n\} \]

\[ E = \{v_iv_{i+1}/1 \leq i \leq m-1\} \cup v_nv_n \cup u_1v_1 \cup \{u_iu_{i+1}/1 \leq i \leq n - 1\}. \]

The procedure for dominator coloring of vertices is explained below

**Case 1:** When $m = 4$ and $1 \leq n \leq 5$
Let the vertex $v_1$ and $v_3$ be given color 1 and 3 respectively. The vertices $v_2$ and $v_4$ are given color 2. Let the vertices $u_{2j-1}$, for $1 \leq j \leq \lceil \frac{n}{2} \rceil$ be assigned color 3 and the vertices $u_{2j}$, $1 \leq j \leq \lceil \frac{n-1}{2} \rceil$ be assigned color $3+j$ respectively.

Then the vertex $v_1$ and adjacent vertices of $v_1$ dominate color class 1. The vertex $v_3$ dominates color class 2. For $2 \leq j \leq 5$, the vertices $u_j$ dominate color class $3 + \lceil \frac{j-1}{2} \rceil$.

Since every adjacent vertex is given different color it is a proper coloring of vertices and every vertex of the graph dominates all vertices of at least one color class. Therefore this is a dominator coloring of vertices with dominator chromatic number 

$$\chi_d(D(m,n)) = 3 + \lceil \frac{n-1}{2} \rceil, \text{ for } m = 4, \ 1 \leq n \leq 5.$$

*Figure 1*: Dominator chromatic number of dragon graph $D(4, 5)$ is 5. i.e., $\chi_d(D(4, 5)) = 5$

**Case 2**: When $m = 4$ and $n \geq 6$

The vertex $v_1$ and $v_3$ are given color 1 and 3 respectively. The vertices $v_2$ and $v_4$ are given color 2. Let the end vertex $u_n$ of the path $P_n$ be assigned the color 3, when $n \equiv 1 \pmod{3}$. Otherwise it is given color $\lceil \frac{n}{3} \rceil + 4$. Then the remaining uncolored vertices $u_j$ of the path $P_n$ for $1 \leq j \leq n-1$ are assigned color 3 when $j \equiv 1 \pmod{3}$ or color 4 when $j \equiv 2 \pmod{3}$ or color $\lceil \frac{j}{3} \rceil + 4$ when $j \equiv 0 \pmod{3}$.

Then the vertices $v_1$ and adjacent vertices of $v_1$ dominate color class 1. The vertex $v_3$ dominates color class 2. The vertices $u_j$ for $2 \leq j \leq n$, dominate color class $\lceil \frac{j-1}{3} \rceil + 4$.

Since every adjacent vertex is given different color it is a proper coloring of vertices and every vertex of the graph dominates all vertices of at least one color class. Therefore this is a dominator coloring of vertices with dominator chromatic number...
\( \chi_d(D(m, n)) = 4 + \left\lceil \frac{n-1}{3} \right\rceil \), when \( m = 4 \) and \( n \geq 6 \).

**Figure 2:** Dominator chromatic number of dragon graph \( D(4, 9) \) is 7. i.e.,
\[ \chi_d(D(4, 9)) = 7 \]

**Case 3:** When \( m \neq 4 \) and \( \forall n \)

Let the vertex \( v_1 \) be given color 1. Then the vertices \( v_i \) for \( 2 \leq i \leq m \) of the cycle graph \( C_m \) are assigned the color 3 when \( i \equiv 0 \pmod{3} \) or color 2 when \( i \equiv 2 \pmod{3} \) or color \( \left\lceil \frac{i+8}{3} \right\rceil \) when \( i \equiv 1 \pmod{3} \). Let the end vertex \( u_n \) of the path \( P_n \) be assigned the color 2 when \( n \equiv 1 \pmod{3} \). Otherwise it is given color \( \left\lceil \frac{m+6}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil \). After the vertex \( u_n \), the remaining uncolored vertices \( u_j \) of the path \( P_n \) for \( 1 \leq j \leq n-1 \) are assigned color 2 when \( j \equiv 1 \pmod{3} \) or color \( \left\lceil \frac{m+8}{3} \right\rceil + \left\lceil \frac{j-2}{3} \right\rceil \) when \( j \equiv 0 \pmod{3} \).

Then the vertex \( v_1 \) and adjacent vertices of \( v_1 \) (i.e., \( v_2, v_n \), and \( u_1 \)) dominate color class 1. For \( 3 \leq i \leq m-1 \) the vertices \( v_i \) dominate color class \( \left\lceil \frac{m+6}{3} \right\rceil + 3 \). For \( 2 \leq j \leq n-1 \) the vertices \( u_j \) dominate color class \( \left\lceil \frac{m+8}{3} \right\rceil + \left\lceil \frac{j-2}{3} \right\rceil \).

Since every adjacent vertex is given different color it is a proper coloring of vertices and every vertex of the graph dominates all vertices of at least one color class. Therefore this is a dominator coloring of vertices. Thus the dominator chromatic number of dragon graph is given by

\[ \chi_d(D(m, n)) = \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil + 2 \] for any \( n \) and \( m \neq 4 \).

**Figure 3:** Dominator chromatic number of dragon graph \( D(6, 9) \) is 7. i.e.,
\[ \chi_d(D(6, 9)) = 7 \]
Proposition 2.2: The chromatic number of a dragon graph $D(m,n)$ which is obtained by joining a cycle graph $C_m$ to a path graph $P_n$ by a bridge is given by

$$\chi(D(m,n)) = \begin{cases} 2 & \text{if } m \text{ is even} \\ \frac{3}{2} & \text{if } m \text{ is odd} \end{cases}$$

(2)

Lemma 2.3: The domination number of a dragon graph $D(m,n)$, $m \geq 3$ and $\forall n$, is given by

$$\gamma(D(m,n)) = \begin{cases} 2 + \left\lceil \frac{n}{3} \right\rceil & \text{if } m = 4 \text{ and } \forall n \\ \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil & \text{otherwise} \end{cases}$$

(3)

Proof:

Case 1: When $m=4$

If $n \pmod{3} \equiv 0$, then $S = \{v_3\} \cup \{u_n\} \cup \{u_i/1 \leq i \leq n \text{ and } i \pmod{3} \equiv 1\}$. Otherwise $S = \{v_3\} \cup \{u_i/1 \leq i \leq n \text{ and } i \pmod{3} \equiv 1\}$. Clearly every vertex in $V - S$ has at least a neighbor in $S$. Also $S$ is the smallest dominating set. Hence the domination number of Dragon graph $D(4, n)$ is given by $2 + \left\lceil \frac{n}{3} \right\rceil$.

Case 2: When $m \neq 4$

If $n \pmod{3} \equiv 2$, then $S = \{v_i/1 \leq i \leq m \text{ and } i \pmod{3} \equiv 1\} \cup \{u_n\} \cup \{u_i/1 \leq i \leq n \text{ and } i \pmod{3} \equiv 0\}$.

Otherwise $S = \{v_i/1 \leq i \leq m \text{ and } i \pmod{3} \equiv 1\} \cup \{u_i/1 \leq i \leq n \text{ and } i \pmod{3} \equiv 0\}$. Clearly every vertex in $V - S$ has at least a neighbor in $S$. Also $S$ is the smallest dominating set. Hence the domination number of Dragon graph $D(m, n)$ is given by $\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil$.

Hence the domination number of dragon graph $D(m,n)$, $m \geq 3$ and $\forall n$, is given by
\[ \gamma \left( (D(m,n)) \right) = \begin{cases} 
2 + \left\lceil \frac{n}{3} \right\rceil & \text{if } m = 4 \text{ and } \forall n \\
\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil & \text{otherwise} 
\end{cases} \]

**Theorem 2.4:** For \( m \geq 3 \) and \( \forall n \), the dragon graph \( D(m, n) \), satisfies the relation
\[
\chi_d(D(m,n)) = \begin{cases} 
\gamma + \chi - 1 & \text{if } m = 4 \text{ and } n = 1,3 \\
\gamma + \chi + 1 & \text{if } m = 4, n \geq 6 \text{ and } n \pmod{3} \equiv 2(4) \\
\gamma + \chi & \text{otherwise} 
\end{cases}
\]

Proof follows from theorem 2.1, proposition 2.2 and lemma 2.3.

### 3. DOMINATOR CHROMATIC NUMBER OF LOLLIPPOP GRAPH

**Theorem 3.1:** The dominator chromatic number of lollipop graph \( L(m, n) \) when \( m \geq 3 \) and \( \forall n \) is given by
\[
\chi_d(L(m,n)) = m + \left\lceil \frac{n-1}{3} \right\rceil 
\]  \hspace{1cm} (5)

**Proof:**

Consider a lollipop graph \( L(m, n) \) obtained by joining a complete graph \( K_m \) to a path graph \( P_n \) with a bridge. It has \( (m + n) \) vertices and \( \left\lceil \frac{m(m-1)}{2} + n \right\rceil \) edges. The vertex set \( V \) and the edge set \( E \) of Lollipop graph is given by
\[
V = \{v_i/1 \leq i \leq m\} \cup \{u_j/1 \leq j \leq n\} \\
E = \{v_iv_j/1 \leq i \leq m-1, i+1 \leq j \leq m\} \cup u_1v_1 \cup \{u_iu_{i+1}/1 \leq i \leq n-1\}
\]

The following procedure gives a dominator coloring of vertices

Let the vertices \( v_i \) for \( 1 \leq i \leq m \) be given color \( i \) respectively. The end vertex \( u_n \) of the path \( P_n \) is assigned the color \( m \) when \( n \equiv 1(\mod 3) \) else it is given color \( m + \left\lceil \frac{n-1}{3} \right\rceil \). After the end vertex \( u_n \), the remaining uncolored vertices \( u_j \) for \( 1 \leq j \leq n-1 \) are assigned color \( m \) when \( j \equiv 1(\mod 3) \) or color \((m-1)\) when \( j \equiv 2(\mod 3) \) or color \( m + \frac{1}{3} \) when \( j \equiv 0(\mod 3) \)

Then the vertex \( u_1 \) and \( v_i \) for \( 1 \leq i \leq m \) dominates color class 1. Also the vertices \( u_j \) for \( 2 \leq j \leq n \) dominate color class \( m + \left\lceil \frac{n-1}{3} \right\rceil \) respectively. Since every adjacent vertex is given different color it is a proper coloring of vertices and every vertex of the graph dominates all vertices of at least one color class. Therefore this is a dominator coloring and the dominator chromatic number of lollipop graph \( L(m,n) \) for \( m \geq 3 \) and \( \forall n \) is
\[
\chi_d(L(m, n)) = m + \left\lceil \frac{n-1}{3} \right\rceil.
\]
**Figure 4:** Dominator chromatic number of lollipop graph \(L(6, 9)\) is 9, i.e., 
\[\chi_d(L(6, 9)) = 9\]

**Proposition 3.2:** The chromatic number of lollipop graph \(L(m,n)\) obtained by joining a complete graph \(K_m\) to a path graph \(P_n\) by a bridge is given by
\[\chi(L(m,n)) = m\] (6)

**Lemma 3.3:** The domination number of lollipop graph \(L(m,n)\) is given by
\[\gamma(L(m,n)) = \lceil \frac{n-1}{3} \rceil + 1\] (7)

**Proof:**
We know that domination number of complete graph \(K_m\) is 1 and that of a path graph \(P_n\) is given by \(\lceil \frac{n-1}{3} \rceil\). Hence the domination number for lollipop graph \(L(m,n)\) which is obtained by joining a complete graph \(K_m\) to a path graph \(P_n\) with a bridge is given by \(\gamma(L(m,n)) = \lceil \frac{n-1}{3} \rceil + 1\).

**Theorem 3.4:** For \(m \geq 3\) and \(\forall n\), the lollipop graph \(L(m,n)\) satisfies the relation 
\[\chi_d(L(m,n)) = \gamma(L(m,n)) + \chi(L(m,n)) - 1\]

Proof follows from theorem 3.1, proposition 3.2 and lemma 3.3

**REFERENCES**


