

Multiplicative Zagreb indices of some classes of graphs

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Abstract

Let $G = (V, E)$ be a simple graph. Let $v \in V(G)$ and $d_G(v)$ denote the number of edges incident with v . The multiplicative Zagreb indices are defined as $\Pi_1(G) = \prod_{v \in V(G)} d_G(v)^2$ and $\Pi_2(G) = \prod_{uv \in E(G)} d_G(u)d_G(v)$. In this paper we present the upper bounds on the multiplicative version of Zagreb indices of complementary prism and Mycielskian graph of a graph in terms of graph parameters including the size of graph, first zagreb index, order of graph and second Zagreb index. We also determine the multiplicative Zagreb indices of the complementary prism and Mycielskian of some standard graphs.

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1 Introduction

We denote by n the order of a graph containing no loops and multiple edges. The set $V(G)$ and $E(G)$ represents respectively the

vertex set and edge set of graph G . The number of edges of a graph G is called the size of G and is denoted by m . We say that two vertices are adjacent, if there is an edge incident with both the vertices. The degree of a vertex v denoted by $d_G(v)$ is the number of vertices adjacent to v . The first and second Zagreb indices are defined as

$$M_1(G) = \sum_{v \in V(G)} (d_G(v))^2$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v)$$

The reader is referred to [1, 2, 3, 4] for mathematical properties of Zagreb indices. A variant of Zagreb index, the multiplicative version of Zagreb indices was defined in [5, 6] as

$$\Pi_1(G) = \prod_{v \in V(G)} d_G(v)^2$$

and

$$\Pi_2(G) = \prod_{uv \in E(G)} d_G(u)d_G(v).$$

Topological indices are widely used in chemistry and indices are classified based on degree concepts, distance concepts and combination of both degree and distance concepts. In the study of molecular complexity, the Zagreb indices and its variants are widely used. The mathematical properties and applications of multiplicative version of zagreb indices(abbreviated, MZI) are given in [8, 9].

We compute the MZI of two classes of graphs namely Complementary prism and Mycielskian graph of a graph.

2 Preliminaries

To start with, we define complementary prism of a graph and mycielskian graph of a graph and then calculate its multiplicative Zagreb indices.

Definition 1. [10] The complementary prism $G\overline{G}$ of G is the graph formed from the disjoint union of G and $\overline{G}$ by adding the

edges of a perfect matching between the corresponding vertices of G and $\overline{G}$.

Definition 2. [7] The mycielskian graph $\mu(G)$ of $G = (V, E)$ is defined as follows: If $V = \{v_1, v_2, \dots, v_n\}$ then $V(\mu(G)) = V \cup V^1 \cup \{x\}$, where $V^1 = \{u_1, u_2, \dots, u_n\}$ is a copy of V and $E(\mu(G)) = E(G) \cup \{u_i v_j : v_i v_j \in E(G)\} \cup \{x u_i : 1 \leq i \leq n\}$.

Lemma 3. [9] Let $x_1, x_2, \dots, x_n$ be nonnegative numbers. Then $\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \dots x_n}$ holds with equality if and only if $x_1 = x_2 = \dots = x_n$.

3 First Multiplicative Zagreb index of Complementary prism and Mycielskian graph

It is easy to observe the following:

Remark 4. It is to be noted that

$$\deg_{G\overline{G}}(v) = \begin{cases} \deg_G(v) + 1, & \text{if } v \in V(G) \\ \deg_{\overline{G}}(v) + 1, & \text{if } v \in V(\overline{G}) \end{cases}$$

We now find the first multiplicative zagreb index of complementary prisms.

Theorem 5. For any graph G with size m ,

$$\Pi_1(G\overline{G}) \leq \left(\frac{n+4m+M_1(G)}{n} \right)^n \left(\frac{n+4m_1+M_1(\overline{G})}{n} \right)^n,$$

where m_1 is the size of $\overline{G}$. Equality holds if and only if G is regular.

Proof. By definition,

$$\begin{aligned} \Pi_1(G\overline{G}) &= \prod_{v \in V(G)} (d_{G\overline{G}}(v))^2 \prod_{v \in V(\overline{G})} (d_{G\overline{G}}(v))^2 \\ &= \prod_{v \in V(G)} (d_G(v) + 1)^2 \prod_{v \in V(\overline{G})} (d_{\overline{G}}(v) + 1)^2 \end{aligned}$$

from Lemma 3, we get

$$\begin{aligned} &\leq \left[\sum_{v \in V(G)} \frac{(d_G(v))^2 + 2(d_G(v)) + 1}{n} \right]^n \left[\sum_{v \in V(\overline{G})} \frac{(d_{\overline{G}}(v))^2 + 2(d_{\overline{G}}(v)) + 1}{n} \right]^n \quad (A) \\ &\leq \left[\frac{n + 4m + M_1(G)}{n} \right]^n \left[\frac{n + 4m_1 + M_1(\overline{G})}{n} \right]^n \end{aligned}$$

Equality is obtained if and only if

$$(d_G(v))^2 + 2d_G(v) + 1 = (d_G(u))^2 + 2d_G(u) + 1$$

Which gives

$$d_G(v) - d_G(u) = 0$$

Therefore, G is regular. $\square$

Example 6. For the graph $G = C_p$, we have $\Pi_1(C_p \overline{C_p}) = (9(p-2)^2)^p$.

Example 7. For the graph $G = K_p$, we have $\Pi_1(K_p \overline{K_p}) = p^{2p}$.

We can also express the right hand side of Theorem 3 in terms of parameters in G .

Corollary 8. For any graph G ,
 $\Pi_1(G\overline{G}) \leq \left(\frac{M_1(G) + 4m + n}{n} \right)^n \left(\frac{M_1(G) + n^3 - 4nm}{n} \right)^n$.

Proof. Replacing $d_{\overline{G}}(v)$ with $(n-1-d_G(v))$ in equation (A), we get the required inequality. $\square$

We now express the right hand side of Theorem 3 in terms of parameters in $\overline{G}$.

Corollary 9. For any graph G ,
 $\Pi_1(G\overline{G}) \leq \left(\frac{M_1(\overline{G}) + 4m_1 + n}{n} \right)^n \left(\frac{M_1(\overline{G}) + n^3 - 4nm_1}{n} \right)^n$.

Proof. Replacing $d_G(v)$ with $(n-1-d_{\overline{G}}(v))$ in equation (A), we get the required inequality. $\square$

We now compute the first multiplicative zagreb index of Mycielskian graphs.

Remark 10. It is to be noted that

$$\deg_{\mu(G)}(v) = \begin{cases} 2\deg_G(v), & \text{if } v \in V(G) \\ \deg_G(v) + 1, & \text{if } v \in V^1(G) \end{cases}$$

and $d_{\mu(G)}(x) = n$.

Theorem 11. For any graph G , $\Pi_1(\mu(G)) \leq 4^n n^2 \Pi_1(G) \left[\frac{M_1(G) + 4m + n}{n} \right]^n$. Equality holds if and only if G is regular.

Proof. By definition,

$$\begin{aligned} \Pi_1(\mu(G)) &= \prod_{v \in V(G)} d_{\mu(G)}(v) \cdot \prod_{v \in V^1(G)} d_{\mu(G)}(v) \cdot n^2 \\ &\leq \prod (2d_G(v))^2 \cdot \prod (d_G(v) + 1)^2 \cdot n^2 \end{aligned}$$

by Lemma 3,

$$\begin{aligned} &\leq 4^n n^2 \sum \left(\frac{(d_G(v))^2 + 2d_G(v) + 1}{n} \right)^n \\ &\leq 4^n n^2 \Pi_1(G) \left[\frac{M_1(G) + 4m + n}{n} \right]^n \end{aligned}$$

arguing as in Theorem 5, we get the equality condition. $\square$

Example 12. For the cycle C_p , $\Pi_1(\mu(G)) = 144^p \cdot p^2$.

Example 13. For the complete graph K_p , $\Pi_1(\mu(G)) = 4^p p^2 (p-1)^{2p} p^{2p}$.

4 Second Multiplicative Zagreb index of Complementary prism and Mycielskian graph:

We now compute the second multiplicative Zagreb index of Complementary prism.

Theorem 14. For any graph G with size m and $\overline{G}$ with size m_1 , $\Pi_2(G\overline{G}) \leq \left[\frac{M_2(G)+m+M_1(G)}{m} \right]^m \left[\frac{M_2(\overline{G})+m_1+M_1(\overline{G})}{m_1} \right]^{m_1} \left[\frac{4m_1m+n^3}{n^2} \right]^n$. Equality holds if and only if G is regular.

Proof. By definition,

$$\begin{aligned} \Pi_2(G\overline{G}) &= \prod_{uv \in E(G\overline{G})} d_{G\overline{G}}(u)d_{G\overline{G}}(v) \\ &= \prod_{u_i u_j \in E(G)} (d_G(u_i) + 1)(d_G(u_j) + 1) \prod_{v_i v_j \in E(\overline{G})} (d_{\overline{G}}(v_i) + 1)(d_{\overline{G}}(v_j) + 1) \\ &\quad \prod_{u \in V(G), v \in V(\overline{G})} (d_G(u) + 1)(d_{\overline{G}}(v) + 1) \end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} &\leq \left[\sum_{u_i u_j \in E(G)} \frac{(d_G(u_i)d_G(u_j) + d_G(u_i) + d_G(u_j) + 1)}{m} \right]^m X \\ &\quad \left[\sum_{v_i v_j \in E(\overline{G})} \frac{(d_{\overline{G}}(v_i)d_{\overline{G}}(v_j) + d_{\overline{G}}(v_i) + d_{\overline{G}}(v_j) + 1)}{m_1} \right]^{m_1} X \\ &\quad \left[\sum_{u \in V(G), v \in V(\overline{G})} \frac{d_G(u)d_{\overline{G}}(v) + d_{\overline{G}}(v) + d_G(u) + 1}{n^2} \right]^n \\ &\leq \left[\frac{M_2(G) + m + M_1(G)}{m} \right]^m \left[\frac{M_2(\overline{G}) + m_1 + M_1(\overline{G})}{m_1} \right]^{m_1} \\ &\quad \left[\frac{4mm_1 + 2n(m + m_1) + n^2}{n^2} \right]^n \\ &\leq \left[\frac{M_2(G) + m + M_1(G)}{m} \right]^m \left[\frac{M_2(\overline{G}) + m_1 + M_1(\overline{G})}{m_1} \right]^{m_1} \left[\frac{4m_1m + n^3}{n^2} \right]^n \end{aligned}$$

Arguing as in theorem 5, we get the equality. $\square$

Example 15. For the cycle C_p , $\Pi_2(C_p\overline{C_p}) = 27^p(p-2)^{p(p-2)}$.

Theorem 16. For any graph G , $\Pi_2(\mu(G)) \leq 4^m \Pi_2(G) [8m + 4n]^m (2m + n)^n$.

Proof. By definition,

$$\begin{aligned}\Pi_2(\mu(G)) &= \prod_{uv \in E(\mu(G))} d_{\mu(G)}(u)d_{\mu(G)}(v) \\ &= \prod_{uv \in E(G)} (2d_G(u))(2d_G(v)) \prod_{u \in V(G), v \in V^1(G)} (2d_G(u))(d_G(v) + 1) \\ &\quad \prod_{x \in V(\mu(G)), v \in V^1(G)} (d_{\mu(G)}(x))(d_G(v) + 1)\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}&\leq 4^m \Pi_2(G) \left[\sum_{u \in V(G), v \in V^1(G)} \frac{2d_G(u)d_G(v) + 2d_G(u)}{m} \right]^m n^n \left[\sum \frac{d_G(v) + 1}{n} \right]^n \\ &\leq 4^m \Pi_2(G) \left[\frac{2(2m)^2 + 4mn}{m} \right]^m (2m + n)^n \\ &\leq 4^m \Pi_2(G) [8m + 4n]^m (2m + n)^n\end{aligned}$$

□

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