

On gp^{**} -closed set in Topological spaces

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Abstract

The aim of this paper is to introduce a new type of closed set namely gp^{**} -closed set. In this paper, we discuss the relationship between gp^{**} -closed set and the existing closed sets. Further its characteristics are also discussed.

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1 Introduction

The concept of generalized closed set was introduced and discussed by Levine. N in 1970 [1] it includes connectedness, compactness, normal and separation axioms. Followed by Levine others introduce the closed sets like gs -closed set, gsp -closed set, g^*s -closed sets etc. In 1998, H. Maki, J. Umehara and T. Noiri[2] introduced the concept of gp -closed set and pg -closed set. They discussed basic properties, continuous functions and irresolute function. M. K. R. S. Veerakumar[12] introduce g^* -closed set in 1991. J. Dontchev introduced gsp -closed set in 1995. Later in 2013, P. Jayakumar and their team introduced gp^* -closed set in the topological space. They discussed some of their basic properties, and its characteristics. After that in 2014 P. Jayakumar and their team introduced the concept of gp^* -connectedness and gp^* -compactness in topological space and they proved some theorems. In 2015 Priscilla pacifica

and their team introduced the concept of $(gsp)^{**}$ -closed set in topological space. The purpose of this paper is to introduce the concept of gp^{**} closed set in topological space.

2 Preliminaries

Throughout this paper (X, τ) , (Y, σ) and (Z, η) will always denote topological spaces on which no separation axioms are assumed unless or otherwise mentioned. For a subset A of a space (X, τ) , $cl(A)$ and $int(A)$ denotes the closure and interior of A respectively.

We begin by recalling the following definitions which are useful for this paper.

Definition 2.1. A Subset A of topological space (X, τ) is called

- (1) a semiopen set if $A \subseteq cl(int(A))$ and a semi-closed set if $int(cl(A)) \subseteq A$
- (2) a semipre open set if $A \subseteq cl(int(cl(A)))$ and a semi-pre closed set if $int(cl(int(A))) \subseteq A$
- (3) a regular-open set if $int(cl(A)) = A$ and a regular-closed if $cl(int(A)) = A$
- (4) a pre-open set if $A \subseteq int(cl(A))$ and a pre-closed set if $cl(int(A)) \subseteq A$
- (5) an α -open set if $A \subseteq int(cl(int(A)))$ and α -closed set if $cl(int(cl(A))) \subseteq A$.

Definition 2.2. A Subset A of topological space (X, τ) is said to be

- (1) a *generalized closed set* if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (2) a *g^* -closed set* if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open in (X, τ) .

- (3) a g^{**} -closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g^* open in (X, τ) .
- (4) a *generalized pre closed set* (briefly *gp*-closed) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (5) a gp^* -closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is *gp*-open in (X, τ) .
- (6) a αg -closed set if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (7) a αg^{**} -closed set if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g^{**} -open in (X, τ) .
- (8) a $s\alpha g^*$ -closed set if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g^* -open in (X, τ) .
- (9) a *gsp*-closed set if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (10) a sg^* -closed set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is g^* -open in (X, τ) .
- (11) a *gpr*-closed set if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ) .

3 gp^{**} -closed sets and its properties

Definition 3.1. A subset A of a topological space (X, τ) is called gp^{**} -closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is gp^* open.

Example 3.1. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Here gp^{**} -closed sets are $\{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$.

Definition 3.2. A subset S of a topological space (X, τ) is called gp^{**} -open set

Example 3.2. Let $X = Y = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Here gp^{**} -open sets are $\{\phi, X, \{a, c\}, \{c\}, \{a\}\}$.

Theorem 1. A subset S of X is gp^{**} -closed set in $X \iff cl(S) - S$ contains no non-empty gp^* closed set in X .

Proof. Let M be gp^* -closed set of X such that $M \subseteq cl(S) - S$. Then $M \subseteq cl(S) \cap S^c$. Therefore $M \subseteq cl(S)$ and $M \subseteq S^c$. since M^c is gp^* open set and S is gp^{**} -closed set, $cl(S) \subseteq M^c$ (i.e) $M \subseteq cl(S)^c$. Hence $M \subseteq cl(S) \cap cl(S)^c = \phi$ (i.e) $M = \phi$. Then $cl(S) - S$ contains no non empty gp^* closed set.

conversely, Assume that $cl(S) - S$ contains no non-empty gp^* closed set in X . Let $S \subseteq U$, U is gp^* open. Suppose $cl(S)$ is not contained in U . Then $cl(S) \cap U^c$ is a non empty gp^* closed set and contained in $cl(S) - S$, Which is a contradiction. Therefore $cl(S) \subseteq U$. Hence S is gp^{**} closed set. $\square$

Theorem 2. If M is gp^{**} closed set in X and $M \subseteq N \subseteq cl(M)$ then N is gp^{**} closed set in X .

Proof. Let U be gp^* open set in X . since M is gp^{**} closed set in X , we have $cl(M) \subseteq U$, whenever $M \subseteq U$ and U is gp^{**} open set. Let $M \subseteq N \subseteq cl(M) \subseteq U$. since $N \subseteq cl(M)$, we have $cl(N) \subseteq cl(M)$. Then $cl(N) - N \subseteq cl(M) - M \subseteq U$. By the above theorem, $cl(M) - M$ contains no non empty gp^* closed set. Therefore $cl(N) - N$ also contains no non empty gp^* closed set. Hence N is gp^{**} closed set in X . $\square$

Theorem 3. If M is gp^{**} open set in X and $int(M) \subseteq N \subseteq M$ then N is gp^{**} open set in X .

Proof. Suppose that $int(M) \subseteq N \subseteq M$ and M is gp^{**} -open set in $X \implies M^c \subseteq N^c \subseteq cl(M)^c$. Given M^c is gp^{**} -closed set in X . By the above theorem, N^c is gp^{**} -closed set. Therefore N is gp^{**} -open set. $\square$

Theorem 4. Every closed set is gp^{**} -closed set.

Proof. Let M be any closed set. Then $cl(M) = M$. Let $M \subseteq U$ and U be gp^* -open. Then $cl(M) \subseteq U$, whenever $M \subseteq U$ and U is gp^* -open. Hence M is gp^{**} -closed set. $\square$

The converse of the above theorem need not be true as seen in the following example:

Example 3.3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}\}$. Then $M = \{a, c\}$ is a gp^{**} closed set but not a closed set.

Theorem 5. Every gp^{**} -closed set is sg^* -closed set.

Proof. Let M be gp^{**} -closed set in X . Let U be g^* -open set such that $M \subseteq U$. since every g^* -open set is gp^* -open set and M is gp^{**} -closed set, we have, $scl(M) \subseteq cl(M) \subseteq U$, whenever $M \subseteq U$. Hence M is sg^* -closed set. $\square$

The converse of the above theorem need not be true as seen in the following example:

Example 3.4. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Then $M = \{c\}$ is sg^* closed set but not gp^{**} closed set.

Theorem 6. Every gp^{**} -closed set is gsp -closed set.

Proof. Let M be gp^{**} -closed set in X . Let U be an open set such that $M \subseteq U$. since every open set is gp^* -open set and M is gp^{**} -closed set, we have, $spcl(M) \subseteq cl(M) \subseteq U$. Hence M is gsp -closed set. $\square$

The converse of the above theorem need not be true as seen in the following example:

Example 3.5. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Then $M = \{c\}$ is gsp closed set but not gp^{**} closed set.

Theorem 7. Every gp^{**} -closed set is gs -closed set.

Proof. Let M be gp^{**} -closed set in X . Let U be an open set such that $M \subseteq U$. since every open set is gp^* -open set and M is gp^{**} -closed set, we have, $scl(M) \subseteq cl(M) \subseteq U$. Hence M is gs -closed set. $\square$

The converse of the above theorem need not be true as seen in the following example:

Example 3.6. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Then $M = \{c\}$ is gs -closed set but not gp^{**} -closed set.

Theorem 8. Every gp^{**} -closed set is g -closed set.

Proof. Let M be gp^{**} -closed set in X . Let U be an open set such that $M \subseteq U$. since every open set is gp^* open set and M is gp^{**} - closed set, we have, $cl(M) \subseteq U$. Hence M is g -closed set. $\square$

The converse of the above theorem need not be true as seen in the following example:

Example 3.7. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}\}$. Then $M = \{a, c\}$ is g closed set but not gp^{**} closed set.

Theorem 9. Every gp^{**} -closed set is g^{**} -closed set.

Proof. Let M be gp^{**} -closed set in X . Let U be g^* -open set such that $M \subseteq U$. since every g^* -open set is gp^* -open set and M is gp^{**} -closed set, we have, $cl(M) \subseteq U$. Hence M is g^{**} - closed set. $\square$

The converse of the above theorem need not be true as seen in the following example:

Example 3.8. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}\}$. Then $M = \{a, c\}$ is g^{**} -closed set but not gp^{**} -closed set.

Theorem 10. Every gp^{**} -closed set is αg -closed set.

Proof. Let M be gp^{**} -closed set in X . Let $M \subseteq U$ and U be an open set. since every open set is gp^* -open set and M is gp^{**} - closed set, we have, $\alpha cl(M) \subseteq cl(M) \subseteq U$. Hence M is gs -closed set. $\square$

The converse of the above theorem need not be true as seen in the following example:

Example 3.9. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Then $M = \{c\}$ is αg closed set but not gp^{**} closed set.

Theorem 11. Every gp^{**} -closed set is wg -closed set.

Proof. Let us consider, M be gp^{**} -closed set in X . Let $M \subseteq U$ and U be an g^* - open set. since every open set is gp^* -open set and M is gp^{**} - closed set, we have, $cl(int(M)) \subseteq cl(M) \subseteq U$. Hence M is wg -closed set. $\square$

The converse of the above theorem need not be true as seen in the following example:

Example 3.10. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Then $M = \{c\}$ is wg closed set but not gp^{**} closed set.

Theorem 12. Every gp^{**} -closed set is sag^* -closed set.

Proof. Let M be any gp^{**} -closed set in X . Let $M \subseteq U$ and U be g^* -open set. since every g^* -open set is gp^* -open set and M is gp^{**} -closed set, we have, $\alpha cl(M) \subseteq cl(M) \subseteq U$. Hence M is sag^* -closed set. $\square$

The converse of the above theorem need not be true as seen in the following example:

Example 3.11. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Then $M = \{c\}$ is sag^* closed set but not gp^{**} - closed set.

Theorem 13. Every rg-closed set is gp^{**} -closed set.

Proof. Let M be rg-closed set in X . Therefore, $M \subseteq U$ and U be regular open set. since every regular open set is open set and every open set is gp^* -open set. we have, $cl(M) \subseteq U$. Hence M is gp^{**} -closed set. $\square$

The converse of the above theorem need not be true as seen in the following example:

Example 3.12. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Then $M = \{b\}$ is gp^{**} closed set but not rg closed set.

Theorem 14. Every gp^{**} -closed set is gpr-closed set.

Proof. Let M be gp^{**} -closed set in X . Let $M \subseteq U$ and U be regular open set. since every regular open set is open set and every open set is gp^* -open set. we have, $pcl(M) \subseteq U$. Hence M is gpr-closed set. $\square$

The converse of the above theorem need not be true as seen in the following example:

Example 3.13. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, c\}\}$. Then $M = \{c\}$ is gpr-closed set but not gp^{**} -closed set.

proposition 3.1. $cl(M \cup N) = cl(M) \cup cl(N)$, where M and N are gp^{**} closed sets.

Proof. If $x \in cl(M)$, then every neighbourhood of x contains a point of M and therefore $M \cup N$. $\Rightarrow cl(M) \subseteq cl(M \cup N)$. similarly, $cl(N) \subseteq cl(M \cup N)$. Hence

$$cl(M) \cup cl(N) \subseteq cl(M \cup N) \quad (1)$$

If $x \in cl(M \cup N)$, then every neighbourhood of x contains a point of $M \cup N$ and therefore it contains a point of $cl(M) \cup cl(N)$. Hence $x \in cl(M) \cup cl(N) \Rightarrow$

$$cl(M \cup N) \subseteq cl(M) \cup cl(N) \quad (2)$$

from (1) and (2), $cl(M \cup N) = cl(M) \cup cl(N)$ $\square$

Theorem 15. If M and N are gp^{**} closed set in X then $M \cup N$ is gp^{**} closed set in X .

Proof. Let M and N are gp^{**} closed set in X and U be any gp^* open set containing M and N . Therefore $cl(M) \subseteq U$ and $cl(N) \subseteq U$. since $M \subseteq U$, $N \subseteq U$ then $M \cup N \subseteq U$. Hence $cl(M \cup N) = cl(M) \cup cl(N) \subseteq U$. Therefore $M \cup N$ is gp^{**} closed set in X . $\square$

Remark:

The intersection of two gp^{**} closed sets need not be gp^{**} closed.

Example 3.14. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}\}$

Here gp^{**} -closed sets are $\{\phi, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$.
Here $\{a, b\} \cap \{c, a\} = \{a\}$ is not a gp^{**} closed set.

Theorem 16. For $x \in X$ then the set $X - \{x\}$ is gp^{**} -closed set or gp^* -open set.

Proof. Assume that $X - \{x\}$ is not gp^* -open set. So X is the only gp^* -open set containing $X - \{x\}$. (i.e) $cl(X - \{x\}) \subseteq X$. Then $X - \{x\}$ is gp^{**} -closed set in X . $\square$

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