Riesz Almost Lacunary multiple triple sequence spaces of $\Gamma^3$ defined by a Orlicz function

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Abstract

In this paper we introduce a new concept for Riesz almost lacunary $\Gamma^3$ sequence spaces strong $P$– convergent to zero with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of Riesz almost lacunary $\Gamma^3$ sequence spaces and also some inclusion theorems are discussed.

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1 Introduction

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R} (\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [8,9], Esi et al. [1-3], Datta et al. [4], Subramanian et al. [10], Debnath et al. [5] and many others.

A triple sequence $x = (x_{mnn})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnn}|^{\frac{1}{m+n+k}} < \infty.$$ 

The space of all triple analytic sequences are usually denoted by $\Lambda^3$. A triple sequence $x = (x_{mnn})$ is called triple entire sequence if

$$|x_{mnn}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$ 

The space of all triple entire sequences are usually denoted by $\Gamma^3$.

2 Definitions and Preliminaries

2.1 Definition

An Orlicz function (see [6]) is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M (0) = 0$, $M (x) > 0$, for $x > 0$ and $M (x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function $M$ is replaced by $M (x + y) \leq M (x) + M (y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([7]) used the idea of Orlicz function to construct Orlicz sequence space.

2.2 Definition

The four dimensional matrix $A$ is said to be RH-regular if it maps every bounded $P-$ convergent sequence into a $P-$ convergent sequence with the same $P-$ limit. The assumption of boundedness was made because a triple sequence spaces which is $P-$ convergent is not necessarily bounded.
2.3 Definition

A triple sequence $x = (x_{mnk})$ of real numbers is called almost $P-$ convergent to a limit 0 if

$$\lim_{p,q,u \to \infty} \sup_{r,s,t \geq 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} |x_{mnk}|^{1/(m+n+k)} \to 0.$$ 

that is, the average value of $(x_{mnk})$ taken over any rectangle 

$$\{(m,n,k): r \leq m \leq r+p-1, s \leq n \leq s+q-1, t \leq k \leq t+u-1\}$$

tends to 0 as both $p,q$ and $u$ to $\infty$, and this $P-$ convergence is uniform in $i, \ell$ and $j$. Let denote the set of sequences with this property as $\hat{\Gamma}^3$.

2.4 Definition

Let $(q_{rst}), (\overline{q}_{rst}), (\overline{\overline{q}}_{rst})$ be sequences of positive numbers and

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1s} & 0 & \ldots \\ q_{21} & q_{22} & \cdots & q_{2s} & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ q_{r1} & q_{r2} & \cdots & q_{rs} & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \end{bmatrix} = q_{11} + q_{12} + \ldots + q_{rs} \neq 0,$$

$$Q_s = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \cdots & \overline{q}_{1s} & 0 & \ldots \\ \overline{q}_{21} & \overline{q}_{22} & \cdots & \overline{q}_{2s} & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \overline{q}_{r1} & \overline{q}_{r2} & \cdots & \overline{q}_{rs} & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \neq 0,$$

$$Q_t = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & \cdots & \overline{\overline{q}}_{1s} & 0 & \ldots \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & \cdots & \overline{\overline{q}}_{2s} & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & \cdots & \overline{\overline{q}}_{rs} & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \end{bmatrix} = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \ldots + \overline{\overline{q}}_{rs} \neq 0.$$ Then

the transformation is given by

$$T_{rst} = \frac{1}{Q_r Q_s Q_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k |x_{mnk}|^{1/(m+n+k)}$$

is called the
Riesz mean of triple sequence \( x = (x_{mnk}) \). If \( P - \lim_{rst \to (x)} = 0, 0 \in \mathbb{R} \), then the sequence \( x = (x_{mnk}) \) is said to be Riesz convergent to 0. If \( x = (x_{mnk}) \) is Riesz convergent to 0, then we write \( P_R - \lim x = 0 \).

### 2.5 Definition

The triple sequence \( \theta_{i,t,j} = \{(m_i, n, k_j)\} \) is called triple lacunary if there exist three increasing sequences of integers such that

\[
m_0 = 0, \quad h_i = m_i - m_{i-1} \to \infty \quad \text{as} \quad i \to \infty \quad \text{and} \quad n_0 = 0, \quad \ell_i = n_i - n_{i-1} \to \infty \quad \text{as} \quad \ell \to \infty .
\]

\[
k_0 = 0, \quad \ell_j = k_j - k_{j-1} \to \infty \quad \text{as} \quad j \to \infty .
\]

Let \( m_{i,t,j} = m_i n_t k_j, h_{i,t,j} = h_i h_t h_j, \) and \( \theta_{i,t,j} \) is determine by

\[
I_{i,t,j} = \{(m, n, k) : m_{i-1} < m < m_i \text{ and } n_{t-1} < n \leq n_t \text{ and } k_{j-1} < k, k_j \}
\]

\[
q_k = \frac{m_i}{m_i - n_{t-1}}, \quad \overline{q} = \frac{n_t}{n_t - k_{j-1}}, \quad \overline{q} = \frac{k_j}{k_j - 1}.
\]

Using the notations of lacunary sequence and Riesz mean for triple sequences.

\( \theta_{i,t,j} = \{(m_i, n, k_j)\} \) be a triple lacunary sequence and \( q_m \overline{q}_n \overline{q}_k \) be sequences of positive real numbers such that

\[
Q_{m_i} = \sum_{m \in (0, m_i]} P_{m_i} Q_{n_t} = \sum_{n \in (0, n_i]} P_{n_t} Q_{n_i} = \sum_{k \in (0, k_j]} P_{k_j}.
\]

\[
H_i = \sum_{m \in (0, m_i]} P_{m_i}, \quad \overline{H}_i = \sum_{n \in (0, n_i]} P_{n_t}, \quad \overline{H}_j = \sum_{k \in (0, k_j]} P_{k_j}.
\]

Clearly,

\[
H_i = Q_{m_i} - Q_{m_{i-1}}, \quad \overline{H}_i = Q_{n_t} - Q_{n_{t-1}}, \quad \overline{H}_j = Q_{k_j} - Q_{k_{j-1}}.
\]

If the Riesz transformation of triple sequences is RH-regular, and \( H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty \) as \( i \to \infty \), \( \overline{H}_i = \sum_{n \in (0, n_i]} P_{n_t} \to \infty \) as \( \ell \to \infty \), \( \overline{H}_j = \sum_{k \in (0, k_j]} P_{k_j} \to \infty \) as \( j \to \infty \), then \( \theta'_{i,t,j} = \{(m_i, n, k_j)\} = \{(Q_m Q_n Q_k)\} \) is a triple lacunary sequence. If the assumptions \( Q_r \to \infty \) as \( r \to \infty \), \( \overline{Q}_s \to \infty \) as \( s \to \infty \) and \( \overline{Q}_t \to \infty \) as \( t \to \infty \) may be not enough to obtain the conditions \( H_i \to \infty \) as \( i \to \infty \), \( \overline{H}_i \to \infty \) as \( \ell \to \infty \) and \( \overline{H}_j \to \infty \) as \( j \to \infty \) respectively. For any lacunary sequences \( (m_i), (n_t) \) and \( (k_j) \) are integers.

Throughout the paper, we assume that \( Q_r = q_{11} + q_{12} + \ldots + q_{rs} \to \infty \) \( (r \to \infty) \), \( \overline{Q}_s = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \to \infty \) \( (s \to \infty) \), \( \overline{Q}_t = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{ts} \to \infty \) \( (t \to \infty) \), such that \( H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty \) as \( i \to \infty \), \( \overline{H}_i = Q_{n_t} - Q_{n_{t-1}} \to \infty \) as \( \ell \to \infty \) and \( \overline{H}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty \) as \( j \to \infty \).
Let $Q_{m,n,k} = Q_m Q_n Q_k$, $H_{i,j} = H_i H_j H_k$, $I'_{i,j} = (m, n, k) : Q_{mi-1} < m < Q_{mi}$, $Q_{n,j-1} < n < Q_{n,j}$ and $Q_{k,j-1} < k < Q_{k,j}$.

Let $Q_{m,i,j} = Q_m Q_{i,j}$, $V_{i,j} = V_i Q_{i,j}$ and $V_{k,j} = V_k Q_{k,j}$.

If we take $q_m = 1, q_n = 1$ and $q_k = 1$ for all $m, n$ and $k$ then $H_{i,j}, Q_{i,j}, V_{i,j}$ and $I'_{i,j}$ reduce to $h_{i,j}, q_{i,j}, v_{i,j}$ and $I_{i,j}$.

Let $f$ be an Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence spaces:

$[\Gamma_{r,s,t}^{3}, q_{r,s,t}, q, p] = \{ P - \lim_{m+n+k \to \infty} \frac{1}{Q_{r,s,t}} \sum_{i \in I_{r,s,t}} \sum_{j \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} q_{m+n+k} q_{n} q_{k} [f(x_{m+n+k,j})]^{p_{mnk}} = 0 \}$, uniformly in $i$, $\ell$ and $j$.

$[\Lambda_{r,s,t}^{3}, q_{r,s,t}, q, p] = \{ x = (x_{mnk}) : P - \sup_{m+n+k \to \infty} \frac{1}{Q_{r,s,t}} \sum_{i \in I_{r,s,t}} \sum_{j \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} q_{m+n+k} q_{n} q_{k} [f(x_{m+n+k,j})]^{p_{mnk}} < \infty \}$, uniformly in $i$, $\ell$ and $j$.

Let $f$ be an Orlicz function, $p = p_{mnk}$ be any factorable double sequence of strictly positive real numbers and $q_m$, $q_n$, and $q_k$ be sequences of positive numbers and $Q_r = q_{11} + \cdots + q_{rs}$, $Q_s = q_{1s} + \cdots + q_{rs}$ and $\bar{Q}_r = \bar{q}_{11} + \cdots + \bar{q}_{rs}$.

If we choose $q_m = 1, q_n = 1$ and $q_k = 1$ for all $m, n$ and $k$, then we obtain the following sequence spaces:

$[\Gamma_{r,s,t}^{3}, q_{r,s,t}, q, p] = \{ P - \lim_{m+n+k \to \infty} \frac{1}{Q_{r,s,t}} \sum_{i \in I_{r,s,t}} \sum_{j \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} q_{m+n+k} q_{n} q_{k} [f(x_{m+n+k,j})]^{p_{mnk}} = 0 \}$, uniformly in $i$, $\ell$ and $j$.

$[\Lambda_{r,s,t}^{3}, q_{r,s,t}, q, p] = \{ P - \sup_{m+n+k \to \infty} \frac{1}{Q_{r,s,t}} \sum_{i \in I_{r,s,t}} \sum_{j \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} q_{m+n+k} q_{n} q_{k} [f(x_{m+n+k,j})]^{p_{mnk}} < \infty \}$, uniformly in $i$, $\ell$ and $j$.

### 2.6 Definition

Let $f$ be an Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence space:

$\theta_{i,j} = \{ (m_i, n_i, k_i) \}$ be a triple lacunary sequence $\Gamma_{f}^{3} [AC_{\theta_{i,j}}, p] = \{ P - \lim_{m+n+k \to \infty} \frac{1}{Q_{r,s,t}} \sum_{i \in I_{r,s,t}} \sum_{j \in I_{r,s,t}} \sum_{k \in I_{r,s,t}} q_{m+n+k} [f(x_{m+n+k,j})]^{p_{mnk}} = 0 \}$, uniformly in $i$, $\ell$ and $j$.

We shall denote $\Gamma_{f}^{3} [AC_{\theta_{i,j}}, p]$ as $\Gamma_{f}^{3} [AC_{\theta_{i,j}}, p]$ respectively when $p_{mnk} = 1$ for all $m, n$ and $k$ If $x$ is in $\Gamma_{f}^{3} [AC_{\theta_{i,j}}, p]$, we shall say that $x$ is almost lacunary $\Gamma_{f}^{3}$ strongly $p$–convergent with respect to the Orlicz function $f$. Also note if $f(x) = x, p_{mnk} = 1$ for
all \( m, n \) and \( k \) then \( \Gamma_f^3 [AC_{\theta_i,\ell,j}, p] = \Gamma_f^3 [AC_{\theta_i,\ell,j}] \) which are defined as follows:

\[
\Gamma_f^3 [AC_{\theta_i,\ell,j}] = \{ P - \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} f (|x_m + i,n + \ell,k + j|)^{1/m+n+k} = 0 \}, \text{ uniformly in } i, \ell \text{ and } j.
\]

Again note if \( p_{mnk} = 1 \) for all \( m, n \) and \( k \) then \( \Gamma_f^3 [AC_{\theta_i,\ell,j}, p] = \Gamma_f^3 [AC_{\theta_i,\ell,j}] \).

2.7 Definition

Let \( f \) be an Orlicz function \( p = (p_{mnk}) \) be any factorable triple sequence of strictly positive real numbers, we define the following sequence space: \( \Gamma_f^3 [p] = \{ P - \lim_{r,s,t \to \infty} \frac{1}{rst} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} f (|x_m + i,n + \ell,k + j|)^{1/m+n+k} p_{mnk} = 0 \}, \text{ uniformly in } i, \ell \text{ and } j. \)

If we take \( f(x) = x, p_{mnk} = 1 \) for all \( m, n \) and \( k \) then \( \Gamma_f^3 [p] = \Gamma_f^3. \)

2.8 Definition

Let \( \theta_{i,\ell,j} \) be a triple lacunary sequence; the triple number sequence \( x \) is \( \widehat{S}_{\theta_{i,\ell,j}} - p- \) convergent to 0 then

\[
\widehat{P-lim}_{i,\ell,j} \max_{i,\ell,j} \{ (m, n, k) \in I_{i,\ell,j} : f (|x_m + i,n + \ell,k + j - 0|)^{1/m+n+k} \} = 0.
\]

In this case we write \( \widehat{S}_{\theta_{i,\ell,j}} - \lim f (|x_m + i,n + \ell,k + j - 0|)^{1/m+n+k} = 0. \)

3 Main Results

3.1 Theorem

If \( f \) be any Orlicz function and a bounded factorable positive triple number sequence \( p_{mnk} \) then \( \Gamma_f^3 [AC_{\theta_i,\ell,j}, P] \) is linear space

Proof: The proof is easy. Therefore omit the proof.
3.2 Theorem

For any Orlicz function $f$, we have $\Gamma^3 [AC_{\theta_{1,\ell,j}}] \subseteq \Gamma^3_f [AC_{\theta_{1,\ell,j}}]$.

Proof: Let $x \in \Gamma^3 [AC_{\theta_{1,\ell,j}}]$ so that for each $i, \ell$ and $j$

$$\Gamma^3 [AC_{\theta_{1,\ell,j}}] = \left\{ \lim_{i,j} \frac{1}{h_{ij}} \sum_{m \in I_{i,j}} \sum_{n \in I_{i,j}} \sum_{k \in I_{i,j}} \left( \frac{1}{m+n+k} \right)^{\frac{1}{m+n+k}} \right\}.$$

Since $f$ is continuous at zero, for $\varepsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for every $t$ with $0 \leq t \leq \delta$. We obtain the following,

$$\lim_{i,j} \frac{1}{h_{ij}} \left( h_{ij} \right)^{\frac{1}{m+n+k}} \frac{1}{h_{ij}} \sum_{m \in I_{i,j}} \sum_{n \in I_{i,j}} \sum_{k \in I_{i,j}} \left( \frac{1}{m+n+k} \right)^{\frac{1}{m+n+k}}.$$

Then for $x \in \Gamma^3_f (P)$, we can write for each $r, s$ and $u$.

$$B_{i,j} = \frac{1}{h_{ij}} \sum_{m \in I_{i,j}} \sum_{n \in I_{i,j}} \sum_{k \in I_{i,j}} \left( \frac{1}{m+n+k} \right)^{\frac{1}{m+n+k}}.$$

3.3 Theorem

Let $\theta_{1,\ell,j} = \{m_i, n_i, k_j\}$ be a triple lacunary sequence with $\liminf \theta_i > 0$, $\liminf \theta_i f > 0$ and $\liminf \theta_i \geq 1$ then for any Orlicz function $f$, $\Gamma^3_f (P) \subset \Gamma^3 [AC_{\theta_{1,\ell,j}}, P]$.

Proof: Suppose $\liminf \theta_i > 0$, $\liminf \theta_i f > 0$ and $\liminf \theta_i \geq 1$ then there exists $\delta > 0$ such that $\theta_i > 1 + \delta$, $\theta_i f > 1 + \delta$ and $\theta_i \geq 1 + \delta$. This implies $\frac{h_i}{m_i} \geq \frac{\delta}{1+\delta}$, $\frac{h_i}{n_i} \geq \frac{\delta}{1+\delta}$ and $\frac{h_i}{k_i} \geq \frac{\delta}{1+\delta}$. Then for $x \in \Gamma^3_f (P)$, we can write for each $r, s$ and $u$.
\[
m_{k-1} \left( \frac{1}{n_{k-1}} \sum_{k'=1}^{k_j} \sum_{n=n_{k-1}+1}^{n_{k+1}} \sum_{m=1}^{m_{k-1}} f \left( \left| \frac{x_{m+i,n+k+j}}{m+n+k} \right| \right)^{p_{mnk}} \right).
\]

Since \( x \in \Gamma_3^j (P) \) the last three terms tend to zero uniformly in \( m, n, k \) in the sense, thus, for each \( r, s \) and \( u \)

\[
B_{i,\ell,j} = \frac{m_{k-1}n_{k-1}}{h_{i,\ell,j}} \left( \frac{1}{n_{k-1}} \sum_{m=1}^{m_{k-1}} \sum_{n=1}^{n_{k-1}} \sum_{k'=1}^{k_j} f \left( \left| \frac{x_{m+i,n+k+j}}{m+n+k} \right| \right)^{p_{mnk}} \right) - \frac{m_{k-1}n_{k-1}k_{j-1}}{h_{i,\ell,j}} \left( \frac{1}{n_{k-1}} \sum_{m=1}^{m_{k-1}} \sum_{n=1}^{n_{k-1}} \sum_{k'=1}^{k_{j-1}} f \left( \left| \frac{x_{m+i,n+k+j}}{m+n+k} \right| \right)^{p_{mnk}} \right) + O(1).
\]

Since \( h_{i,\ell,j} = m_{i-1}n_{k-1} - m_{i-1}n_{k-1-1}k_{j-1} \) we are granted for each \( i, \ell \) and \( j \) the following

\[
\frac{m_{i-1}n_{k-1}}{h_{i,\ell,j}} \leq \frac{1+\delta}{\delta} \quad \text{and} \quad \frac{m_{i-1}n_{k-1-1}k_{j-1}}{h_{i,\ell,j}} \leq \frac{1}{\delta}.
\]

The terms

\[
\left( \frac{1}{n_{k-1}} \sum_{m=1}^{m_{k-1}} \sum_{n=1}^{n_{k-1}} \sum_{k'=1}^{k_j} f \left( \left| \frac{x_{m+i,n+k+j}}{m+n+k} \right| \right)^{p_{mnk}} \right)
\]

and

\[
\left( \frac{1}{n_{k-1}} \sum_{m=1}^{m_{k-1}} \sum_{n=1}^{n_{k-1}} \sum_{k'=1}^{k_{j-1}} f \left( \left| \frac{x_{m+i,n+k+j}}{m+n+k} \right| \right)^{p_{mnk}} \right)
\]

dual gait sequences for each \( i, \ell \) and \( j \). Thus \( B_{i,\ell,j} \) is a gait sequence for each \( i, \ell \) and \( j \). Hence \( x \in \Gamma_3^j \left( AC_{\theta,\ell,j}, P \right) \).

### 3.4 Theorem

Let \( \theta_{i,\ell,j} = \{ m, n, k \} \) be a triple lacunary sequence with \( \limsup_{q \to \infty} q_i < \infty \) and \( \limsup_{q \to \infty} \bar{q}_i < \infty \) then for any Orlicz function \( f, \quad \Gamma_3^j \left( AC_{\theta,\ell,j}, P \right) \subset \Gamma_3^j (P) \).

**Proof:** Since \( \limsup_{q \to \infty} q_i < \infty \) and \( \limsup_{q \to \infty} \bar{q}_i < \infty \) there exists \( H > 0 \) such that \( q_i < H, \quad \bar{q}_i < H \) and \( q_j < H \) for all \( i, \ell \) and \( j \). Let \( x \in \Gamma_3^j \left( AC_{\theta,\ell,j}, P \right) \). Also there exist \( i_0 > 0, \ell_0 > 0 \) and \( j_0 > 0 \) such that for every \( a \geq i_0, b \geq \ell_0 \) and \( c \geq j_0 \) and \( i, \ell \) and \( j \),

\[
A'_{a,b,c} = \frac{1}{n_{a,b,c}} \sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}} f \left( \left| \frac{x_{m+i,n+k+j}}{m+n+k} \right| \right)^{p_{mnk}} \to \lim_{m,n,k \to \infty}.
\]

Let \( G' = \max \{ A'_{a,b,c} : 1 \leq a \leq i_0, 1 \leq b \leq \ell_0 \text{ and } 1 \leq c \leq j_0 \} \) and \( p, q \) and \( t \) be such that \( m_{i-1} < p \leq m_i, \quad n_{\ell-1} < q \leq n_\ell \text{ and } m_{j-1} < t \leq m_j \). Thus we obtain the following:

\[
\frac{1}{p q t} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} \left( \left| \frac{x_{m+i,n+k+j}}{m+n+k} \right| \right)^{p_{mnk}} \leq \frac{1}{m_{i-1}n_{\ell-1}k_{j-1}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} \left( \left| \frac{x_{m+i,n+k+j}}{m+n+k} \right| \right)^{p_{mnk}}
\]
\[
\leq \frac{1}{m_i-k_j} \sum_{i=1}^{m_i} \sum_{\ell=1}^{n_\ell} \sum_{j=1}^{n_j} \left( \sum_{m \in I_a,b,c} \sum_{n \in I_a,b,c} \sum_{k \in I_a,b,c} \left[ \left( |x_{m+i,n+\ell,k+j}| \right)^{1/m+n+k} \right] \right)_{mnk} \]
\[
= \frac{1}{m_i-k_j} \sum_{i=1}^{m_i} \sum_{\ell=1}^{n_\ell} \sum_{j=1}^{n_j} h_{b,c} A_{a,b,c} + \frac{1}{m_i-k_j} \sum_{i=1}^{m_i} \sum_{\ell=1}^{n_\ell} \sum_{j=1}^{n_j} h_{b,c} A_{a,b,c}.
\]
\[
\leq \frac{\varepsilon}{m_i-k_j} \sum_{i=1}^{m_i} \sum_{\ell=1}^{n_\ell} \sum_{j=1}^{n_j} \left( \sup_{a \geq 0, b \geq 0, c \geq 0} A_{a,b,c} \right) \frac{1}{m_i-k_j} \sum_{i=1}^{m_i} \sum_{\ell=1}^{n_\ell} \sum_{j=1}^{n_j} h_{b,c}.
\]
\[
\leq \frac{\varepsilon}{m_i-k_j} \sum_{i=1}^{m_i} \sum_{\ell=1}^{n_\ell} \sum_{j=1}^{n_j} \left[ \left( |x_{m+i,n+\ell,k+j}| - 0 \right)^{1/m+n+k} \right]_{mnk} = 0, \text{ uniformly in } i, \ell \text{ and } j.
\]

Hence \( x \in \Gamma^3(P) \).

### 3.5 Theorem

Let \( (x_{mnk}) \) be a triple lacunary sequence then

(i) \((x_{mnk}) \rightarrow \Gamma^3(\Theta_{m,n,k})\)

(ii) \((AC_{\Theta_{m,n,k}})\) is a proper subset of \((\Theta_{m,n,k})\)

(iii) If \( x \in \Lambda^3 \) and \((x_{mnk}) \rightarrow \Gamma^3(\Theta_{m,n,k})\) then \((x_{mnk}) \rightarrow \Gamma^3(AC_{\Theta_{m,n,k}})\)

(iv) \( \Gamma^3(\Theta_{m,n,k}) \cap \Lambda^3 = \Gamma^3[\text{AC}_{\Theta_{m,n,k}}] \cap \Lambda^3 \).

**Proof:** (i) Since for all \( i, \ell \text{ and } j \)
\[
\left\{ (m,n,k) \in I_{i,\ell,j} : (|x_{m+i,n+\ell,k+j}| - 0)^{1/m+n+k} = 0 \right\} \leq \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} \left( |x_{m+i,n+\ell,k+j}| - 0 \right)^{1/m+n+k} \leq \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} \left( |x_{m+i,n+\ell,k+j}| - 0 \right)^{1/m+n+k} \text{, for all } i, \ell \text{ and } j.
\]

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This implies that for all $i, \ell$ and $j$

$$P - \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0$$

$$(x_{mnk}) = \left[ \begin{array}{cccccccc}
1 & 2 & 3 & \vdots & \sqrt{h_{i,\ell,j}}^{m+n+k} & 0 & \ldots \\
1 & 2 & 3 & \vdots & \sqrt{h_{i,\ell,j}}^{m+n+k} & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \sqrt{h_{i,\ell,j}}^{m+n+k} & 0 & \ldots \\
0 & 0 & 0 & \ldots & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \sqrt{h_{i,\ell,j}}^{m+n+k} & 0 & \ldots \\
\end{array} \right];$$

Here $x$ is an trible sequence and for all $i, \ell$ and $j$

$$P - \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left\{ (m, n, k) \in I_{i,\ell,j} : (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0 \right\} = 0.$$
Suppose \( x \in \Lambda^3 \) then for all \( i, \ell \) and \( j \), \( \left| x_{m+i, n+\ell, k+j} - 0 \right|^{1/m+n+k} \leq M \) for all \( m, n, k \). Also for given \( \epsilon > 0 \) and \( i, \ell \) large for all \( i, \ell \) and \( j \) we obtain the following:

\[
\frac{1}{n_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} \left( \left| x_{m+i, n+\ell, k+j} - 0 \right|^{1/m+n+k} \right) \\
= \frac{1}{n_{i,\ell,j}} \sum_{m \in I_{i,\ell,\ell}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j} \cap \{x_{m+i, n+\ell, k+j} = 0\}} \left( \left| x_{m+i, n+\ell, k+j} - 0 \right|^{1/m+n+k} \right) + \\
\frac{1}{n_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j} \cap \{x_{m+i, n+\ell, k+j} \geq 0\}} \left( \left| x_{m+i, n+\ell, k+j} - 0 \right|^{1/m+n+k} \right)
\]

Therefore \( x \in \Lambda^3 \) and \( (x_{mnk}) \rightarrow \Gamma^3 \left( \hat{S}_{\theta, i, j} \right) \) then \( (x_{mnk}) \rightarrow \Gamma^3 \left( AC_{\theta, i, j} \right) \).

(iv) \( \Gamma^3 \left( \hat{S}_{\theta, i, j} \right) \cap \Lambda^3 = \Gamma^3 \left[ AC_{\theta, i, j} \right] \cap \Lambda^3 \). follows from (i),(ii) and (iii).

### 3.6 Theorem

If \( f \) be any Orlicz function then \( \Gamma^3 \left[ AC_{\theta, i, j} \right] \notin \Gamma^3 \left( \hat{S}_{\theta, i, j} \right) \).

**Proof:** Let \( x \in \Gamma^3 \left[ AC_{\theta, i, j} \right] \), for all \( i, \ell \) and \( j \).

Therefore we have

\[
\frac{1}{n_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} f \left( \left| x_{m+i, n+\ell, k+j} - 0 \right|^{1/m+n+k} \right) \\
\geq \frac{1}{n_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j} \cap \{x_{m+i, n+\ell, k+j} = 0\}} f \left( \left| x_{m+i, n+\ell, k+j} - 0 \right|^{1/m+n+k} \right) + \\
\frac{1}{n_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j} \cap \{x_{m+i, n+\ell, k+j} \geq 0\}} f \left( \left| x_{m+i, n+\ell, k+j} - 0 \right|^{1/m+n+k} \right)
\]

Hence \( x \notin \Gamma^3 \left( \hat{S}_{\theta, i, j} \right) \).

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### References


