

Riesz Almost Lacunary multiple triple sequence spaces of Γ^3 defined by a Orlicz function

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Abstract

In this paper we introduce a new concept for Riesz almost lacunary Γ^3 sequence spaces strong P -convergent to zero with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of Riesz almost lacunary Γ^3 sequence spaces and also some inclusion theorems are discussed.

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1 Introduction

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al.* [8,9], *Esi et al.* [1-3], *Datta et al.* [4], *Subramanian et al.* [10], *Debnath et al.* [5] and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3 . A triple sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty.$$

The space of all triple entire sequences are usually denoted by Γ^3

2 Definitions and Preliminaries

2.1 Definition

An Orlicz function ([see [6]]) is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([7]) used the idea of Orlicz function to construct Orlicz sequence space.

2.2 Definition

The four dimensional matrix A is said to be RH-regular if it maps every bounded P -convergent sequence into a P -convergent sequence with the same P -limit. The assumption of boundedness was made because a triple sequence spaces which is P -convergent is not necessarily bounded.

2.3 Definition

A triple sequence $x = (x_{mnk})$ of real numbers is called almost P -convergent to a limit 0 if

$$\lim_{p,q,u \rightarrow \infty} \sup_{r,s,t \geq 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} |x_{mnk}|^{1/m+n+k} \rightarrow 0.$$

that is, the average value of (x_{mnk}) taken over any rectangle $\{(m, n, k) : r \leq m \leq r+p-1, s \leq n \leq s+q-1, t \leq k \leq t+u-1\}$ tends to 0 as both p, q and u to ∞ , and this P -convergence is uniform in i, ℓ and j . Let denote the set of sequences with this property as $\left[\widehat{\Gamma^3}\right]$.

2.4 Definition

Let $(q_{rst}), (\overline{q_{rst}}), (\overline{\overline{q_{rst}}})$ be sequences of positive numbers and

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1s} & 0\dots \\ q_{21} & q_{22} & \dots & q_{2s} & 0\dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{r1} & q_{r2} & \dots & q_{rs} & 0\dots \\ 0 & 0 & \dots & 0 & 0\dots \end{bmatrix} = q_{11} + q_{12} + \dots + q_{rs} \neq 0,$$

$$\overline{Q}_s = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \dots & \overline{q}_{1s} & 0\dots \\ \overline{q}_{21} & \overline{q}_{22} & \dots & \overline{q}_{2s} & 0\dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{q}_{r1} & \overline{q}_{r2} & \dots & \overline{q}_{rs} & 0\dots \\ 0 & 0 & \dots & 0 & 0\dots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \dots + \overline{q}_{rs} \neq 0,$$

$$\overline{\overline{Q}}_t = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & \dots & \overline{\overline{q}}_{1s} & 0\dots \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & \dots & \overline{\overline{q}}_{2s} & 0\dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & \dots & \overline{\overline{q}}_{rs} & 0\dots \\ 0 & 0 & \dots & 0 & 0\dots \end{bmatrix} = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \dots + \overline{\overline{q}}_{rs} \neq 0. \text{ Then}$$

the transformation is given by

$$T_{rst} = \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^r \sum_{n=1}^s \sum_{k=1}^t q_m \overline{q}_n \overline{\overline{q}}_k |x_{mnk}|^{1/m+n+k} \text{ is called the}$$

Riesz mean of triple sequence $x = (x_{mnk})$. If $P - \lim_{rst} T_{rst}(x) = 0$, $0 \in \mathbb{R}$, then the sequence $x = (x_{mnk})$ is said to be Riesz convergent to 0. If $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - \lim x = 0$.

2.5 Definition

The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$\begin{aligned} m_0 = 0, h_i &= m_i - m_{i-1} \rightarrow \infty \text{ as } i \rightarrow \infty \text{ and} \\ n_0 = 0, \overline{h}_\ell &= n_\ell - n_{\ell-1} \rightarrow \infty \text{ as } \ell \rightarrow \infty. \\ k_0 = 0, \overline{h}_j &= k_j - k_{j-1} \rightarrow \infty \text{ as } j \rightarrow \infty. \end{aligned}$$

Let $m_{i,\ell,j} = m_i n_\ell k_j$, $h_{i,\ell,j} = h_i \overline{h}_\ell \overline{h}_j$, and $\theta_{i,\ell,j}$ is determined by $I_{i,\ell,j} = \{(m, n, k) : m_{i-1} < m < m_i \text{ and } n_{\ell-1} < n \leq n_\ell \text{ and } k_{j-1} < k \leq k_j\}$, $q_k = \frac{m_k}{m_{k-1}}, \overline{q}_\ell = \frac{n_\ell}{n_{\ell-1}}, \overline{q}_j = \frac{k_j}{k_{j-1}}$.

Using the notations of lacunary sequence and Riesz mean for triple sequences.

$\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence and $q_m \overline{q}_n \overline{q}_k$ be sequences of positive real numbers such that

$Q_{m_i} = \sum_{m \in (0, m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n \in (0, n_\ell]} p_{n_\ell}, Q_{k_j} = \sum_{k \in (0, k_j]} p_{k_j}$ and $H_i = \sum_{m \in (0, m_i]} p_{m_i}, \overline{H} = \sum_{n \in (0, n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k \in (0, k_j]} p_{k_j}$. Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}$. If the Riesz transformation of triple sequences is RH-regular, and $H_i = Q_{m_i} - Q_{m_{i-1}} \rightarrow \infty$ as $i \rightarrow \infty, \overline{H} = \sum_{n \in (0, n_\ell]} p_{n_\ell} \rightarrow \infty$ as $\ell \rightarrow \infty, \overline{\overline{H}} = \sum_{k \in (0, k_j]} p_{k_j} \rightarrow \infty$ as $j \rightarrow \infty$, then $\theta'_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \{(Q_{m_i} Q_{n_j} Q_{k_k})\}$ is a triple lacunary sequence. If the assumptions $Q_r \rightarrow \infty$ as $r \rightarrow \infty, \overline{Q}_s \rightarrow \infty$ as $s \rightarrow \infty$ and $\overline{\overline{Q}}_t \rightarrow \infty$ as $t \rightarrow \infty$ may be not enough to obtain the conditions $H_i \rightarrow \infty$ as $i \rightarrow \infty, \overline{H}_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$ and $\overline{\overline{H}}_j \rightarrow \infty$ as $j \rightarrow \infty$ respectively. For any lacunary sequences $(m_i), (n_\ell)$ and (k_j) are integers.

Throughout the paper, we assume that $Q_r = q_{11} + q_{12} + \dots + q_{rs} \rightarrow \infty (r \rightarrow \infty), \overline{Q}_s = \overline{q}_{11} + \overline{q}_{12} + \dots + \overline{q}_{rs} \rightarrow \infty (s \rightarrow \infty), \overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \dots + \overline{\overline{q}}_{rs} \rightarrow \infty (t \rightarrow \infty)$, such that $H_i = Q_{m_i} - Q_{m_{i-1}} \rightarrow \infty$ as $i \rightarrow \infty, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \rightarrow \infty$ as $\ell \rightarrow \infty$ and $\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \rightarrow \infty$ as $j \rightarrow \infty$.

Let $Q_{m_i, n_\ell, k_j} = Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}$, $H_{i\ell j} = H_i \overline{H}_\ell \overline{\overline{H}}_j$,
 $I'_{i\ell j} = (m, n, k) : Q_{m_{i-1}} < m < Q_{m_i}$, $\overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell}$
 and $\overline{\overline{Q}}_{k_{j-1}} < k < \overline{\overline{Q}}_{k_j}$,
 $V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}$, $\overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}}$ and $\overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}}$. and $V_{i\ell j} = V_i \overline{V}_\ell \overline{\overline{V}}_j$.

If we take $q_m = 1$, $\overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all m, n and k then $H_{i\ell j}$, $Q_{i\ell j}$, $V_{i\ell j}$ and $I'_{i\ell j}$ reduce to $h_{i\ell j}$, $q_{i\ell j}$, $v_{i\ell j}$ and $I_{i\ell j}$.

Let f be an Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence spaces:

$$\begin{aligned} [\Gamma_R^3, \theta_{i\ell j}, q, f, p] &= \{P\text{-}\lim_{i, \ell, j \rightarrow \infty} \frac{1}{H_{i, \ell, j}} \sum_{i \in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \\ &[f(|x_{m+i, n+\ell, k+j}|)^{p_{mnk}}] = 0\}, \text{ uniformly in } i, \ell \text{ and } j. \\ [\Lambda_R^3, \theta_{i\ell j}, q, f, p] &= \{x = (x_{mnk}) : P\text{-}\sup_{i, \ell, j} \frac{1}{H_{i, \ell, j}} \sum_{i \in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j \in I_{i\ell j}} \\ &q_m \overline{q}_n \overline{\overline{q}}_k [f(|x_{m+i, n+\ell, k+j}|)^{p_{mnk}}] < \infty, \text{ uniformly in } i, \ell \text{ and } j. \end{aligned}$$

Let f be an Orlicz function, $p = p_{mnk}$ be any factorable double sequence of strictly positive real numbers and $q_m, \overline{q}_n$ and $\overline{\overline{q}}_k$ be sequences of positive numbers and $Q_r = q_{11} + \cdots + q_{rs}$, $\overline{Q}_s = \overline{q}_{11} + \cdots + \overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \cdots + \overline{\overline{q}}_{rs}$,

If we choose $q_m = 1$, $\overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all m, n and k , then we obtain the following sequence spaces.

$$\begin{aligned} [\Gamma_R^3, q, f, p] &= \{P\text{-}\lim_{r, s, t \rightarrow \infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^r \sum_{n=1}^s \sum_{k=1}^t \\ &q_m \overline{q}_n \overline{\overline{q}}_k [f(|x_{m+i, n+\ell, k+j}|)^{p_{mnk}}] = 0\}, \text{ uniformly in } i, \ell \text{ and } j. \\ [\Lambda_R^3, q, f, p] &= \{P\text{-}\sup_{r, s, t} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^r \sum_{n=1}^s \sum_{k=1}^t \\ &q_m \overline{q}_n \overline{\overline{q}}_k [f(|x_{m+i, n+\ell, k+j}|)^{p_{mnk}}] < \infty\}, \text{ uniformly in } i, \ell \text{ and } j. \end{aligned}$$

2.6 Definition

Let f be an Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence space:

$$\begin{aligned} \theta_{i, \ell, j} &= \{(m_i, n_\ell, k_j)\} \text{ be a triple lacunary sequence } \Gamma_f^3 [AC_{\theta_{i, \ell, j}}, p] = \\ &\{P\text{-}\lim_{i, \ell, j} \frac{1}{h_{i, \ell, j}} \sum_{m \in I_{i, \ell, j}} \sum_{n \in I_{i, \ell, j}} \sum_{k \in I_{i, \ell, j}} \\ &\left[f(|x_{m+i, n+\ell, k+j}|)^{1/m+n+k} \right]^{p_{mnk}} = 0\}, \text{ uniformly in } i, \ell \text{ and } j. \end{aligned}$$

We shall denote $\Gamma_f^3 [AC_{\theta_{i, \ell, j}}, p]$ as $\Gamma^3 [AC_{\theta_{i, \ell, j}}, p]$ respectively when $p_{mnk} = 1$ for all m, n and k . If x is in $\Gamma^3 [AC_{\theta_{i, \ell, j}}, p]$, we shall say that x is almost lacunary Γ^3 strongly p -convergent with respect to the Orlicz function f . Also note if $f(x) = x$, $p_{mnk} = 1$ for

all m, n and k then $\Gamma_f^3 [AC_{\theta_{i,\ell,j}}, p] = \Gamma^3 [AC_{\theta_{i,\ell,j}}]$ which are defined as follows:

$$\Gamma^3 [AC_{\theta_{i,\ell,j}}] = \{P - \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} [f(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}] = 0\}, \text{ uniformly in } i, \ell \text{ and } j.$$

Again note if $p_{mnk} = 1$ for all m, n and k then $\Gamma_f^3 [AC_{\theta_{i,\ell,j}}, p] = \Gamma_f^3 [AC_{\theta_{i,\ell,j}}]$. we define

$$\Gamma_f^3 [AC_{\theta_{i,\ell,j}}, p] = \{P - \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} [f(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}} = 0\}, \text{ uniformly in } i, \ell \text{ and } j.$$

2.7 Definition

Let f be an Orlicz function $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence space: $\Gamma_f^3 [p] =$

$$\left\{ P - \lim_{r,s,t \rightarrow \infty} \frac{1}{rst} \sum_{m=1}^r \sum_{n=1}^s \sum_{k=1}^t [f(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}]^{p_{mnk}} = 0 \right\},$$

uniformly in i, ℓ and j .

If we take $f(x) = x, p_{mnk} = 1$ for all m, n and k then $\Gamma_f^3 [p] = \Gamma^3$.

2.8 Definition

Let $\theta_{i,\ell,j}$ be a triple lacunary sequence; the triple number sequence x is $\widehat{S_{\theta_{i,\ell,j}}} - p$ -convergent to 0 then

$$P - \lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \max_{i,\ell,j} \left| \left\{ (m, n, k) \in I_{i,\ell,j} : f(|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} \right\} \right| = 0.$$

In this case we write $\widehat{S_{\theta_{i,\ell,j}}} - \lim (f|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0$.

3 Main Results

3.1 Theorem

If f be any Orlicz function and a bounded factorable positive triple number sequence p_{mnk} then $\Gamma_f^3 [AC_{\theta_{i,\ell,j}}, P]$ is linear space

Proof: The proof is easy. Therefore omit the proof.

3.2 Theorem

For any Orlicz function f , we have $\Gamma^3 [AC_{\theta_{i,\ell,j}}] \subset \Gamma_f^3 [AC_{\theta_{i,\ell,j}}]$

Proof: Let $x \in \Gamma^3 [AC_{\theta_{i,\ell,j}}]$ so that for each i, ℓ and j

$$\Gamma^3 [AC_{\theta_{i,\ell,j}}] =$$

$$\left\{ \lim_{i,\ell,j} \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right] = 0 \right\}.$$

Since f is continuous at zero, for $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for every t with $0 \leq t \leq \delta$. We obtain the following,

$$\begin{aligned} & \frac{1}{h_{i\ell j}} (h_{i\ell j} \varepsilon) + \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j} \text{ and } |x_{m+i,n+\ell,k+j}-0| > \delta} \\ & f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right] \\ & \frac{1}{h_{i\ell j}} (h_{i\ell j} \varepsilon) + \frac{1}{h_{i\ell j}} K \delta^{-1} f(2) h_{i\ell j} \Gamma^3 [AC_{\theta_{i,\ell,j}}]. \end{aligned}$$

Hence i, ℓ and j goes to infinity, we are granted $x \in \Gamma_f^3 [AC_{\theta_{i,\ell,j}}]$.

3.3 Theorem

Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a triple lacunary sequence with $\liminf f_i q_i > 1$, $\liminf f_\ell \bar{q}_\ell > 1$ and $\liminf f_j q_j > 1$ then for any Orlicz function f , $\Gamma_f^3 (P) \subset \Gamma_f^3 (AC_{\theta_{i,\ell,j}}, P)$

Proof: Suppose $\liminf f_i q_i > 1$, $\liminf f_\ell \bar{q}_\ell > 1$ and $\liminf f_j q_j > 1$ then there exists $\delta > 0$ such that $q_i > 1 + \delta$, $\bar{q}_\ell > 1 + \delta$ and $q_j > 1 + \delta$. This implies $\frac{h_i}{m_i} \geq \frac{\delta}{1+\delta}$, $\frac{h_\ell}{n_\ell} \geq \frac{\delta}{1+\delta}$ and $\frac{h_j}{k_j} \geq \frac{\delta}{1+\delta}$. Then for $x \in \Gamma_f^3 (P)$, we can write for each r, s and u .

$$\begin{aligned} B_{i,\ell,j} &= \frac{1}{h_{i\ell j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} = \\ & \frac{1}{h_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} - \\ & \frac{1}{h_{i\ell j}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} - \\ & \frac{1}{h_{i\ell j}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} - \\ & \frac{1}{h_{i\ell j}} \sum_{k=k_j+1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \\ &= \frac{m_i n_\ell k_j}{h_{i\ell j}} \left(\frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \right) - \\ & \frac{m_{i-1} n_{\ell-1} k_{j-1}}{h_{i\ell j}} \left(\frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \right) \\ & - \frac{k_{j-1}}{h_{i\ell j}} \left(\frac{1}{k_{j-1}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \right) \\ & - \frac{n_{\ell-1}}{h_{i\ell j}} \left(\frac{1}{n_{\ell-1}} \sum_{m=m_{k-1}+1}^{m_k} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \right) - \end{aligned}$$

$$\frac{m_{k-1}}{h_{i\ell j}} \left(\frac{1}{m_{k-1}} \sum_{k=1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \right).$$

Since $x \in \Gamma_f^3(P)$ the last three terms tend to zero uniformly in m, n, k in the sense, thus, for each r, s and u

$$B_{i,\ell,j} = \frac{m_i n_\ell k_j}{h_{i\ell j}} \left(\frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \right) - \frac{m_{i-1} n_{\ell-1} k_{j-1}}{h_{i\ell j}} \left(\frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \right) + O(1).$$

Since $h_{i\ell j} = m_i n_\ell k_j - m_{i-1} n_{\ell-1} k_{j-1}$ we are granted for each i, ℓ and j the following

$$\frac{m_i n_\ell k_j}{h_{i\ell j}} \leq \frac{1+\delta}{\delta} \text{ and } \frac{m_{i-1} n_{\ell-1} k_{j-1}}{h_{i\ell j}} \leq \frac{1}{\delta}.$$

The terms $\left(\frac{1}{m_i n_\ell k_j} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \right)$ and $\left(\frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \right)$ are both gai sequences for all i, ℓ and j . Thus $B_{i\ell j}$ is a gai sequence for each i, ℓ and j . Hence $x \in \Gamma_f^3(AC_{\theta_{i,\ell,j}}, P)$.

3.4 Theorem

Let $\theta_{i,\ell,j} = \{m, n, k\}$ be a triple lacunary sequence with $\limsup_{\eta} q_{\eta} < \infty$ and $\limsup_i \bar{q}_i < \infty$ then for any Orlicz function f , $\Gamma_f^3(AC_{\theta_{i,\ell,j}}, P) \subset \Gamma_f^3(p)$.

Proof: Since $\limsup_i q_i < \infty$ and $\limsup_i \bar{q}_i < \infty$ there exists $H > 0$ such that $q_i < H$, $\bar{q}_\ell < H$ and $q_j < H$ for all i, ℓ and j . Let $x \in \Gamma_f^3(AC_{\theta_{i,\ell,j}}, P)$. Also there exist $i_0 > 0$, $\ell_0 > 0$ and $j_0 > 0$ such that for every $a \geq i_0$, $b \geq \ell_0$ and $c \geq j_0$ and i, ℓ and j .

$$A'_{abc} = \frac{1}{h_{abc}} \sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}} f \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty.$$

Let $G' = \max \{A'_{a,b,c} : 1 \leq a \leq i_0, 1 \leq b \leq \ell_0 \text{ and } 1 \leq c \leq j_0\}$ and p, q and t be such that $m_{i-1} < p \leq m_i$, $n_{\ell-1} < q \leq n_\ell$ and $m_{j-1} < t \leq m_j$. Thus we obtain the following:

$$\frac{1}{pqt} \sum_{m=1}^p \sum_{n=1}^q \sum_{k=1}^t \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \leq \frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}}$$

$$\begin{aligned}
&\leq \frac{1}{m_{i-1}n_{\ell-1}k_{j-1}} \sum_{a=1}^i \sum_{b=1}^{\ell} \sum_{c=1}^j \\
&\quad \left(\sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}} \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} \right) \\
&= \frac{1}{m_{i-1}n_{\ell-1}k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c} A'_{a,b,c} + \frac{1}{m_{k-1}n_{\ell-1}k_{j-1}} \\
&\quad \sum_{(i_0 < a \leq i) \cup (\ell_0 < b \leq \ell) \cup (j_0 < c \leq j)} h_{a,b,c} A'_{a,b,c} \\
&\leq \frac{G'}{m_{i-1}n_{\ell-1}k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c} + \frac{1}{m_{i-1}n_{\ell-1}k_{j-1}} \\
&\quad \sum_{(i_0 < a \leq i) \cup (\ell_0 < b \leq \ell) \cup (j_0 < c \leq j)} h_{a,b,c} A'_{a,b,c} \\
&\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{m_{i-1}n_{\ell-1}k_{j-1}} + \frac{1}{m_{i-1}n_{\ell-1}k_{j-1}} \sum_{(i_0 < a \leq i) \cup (\ell_0 < b \leq \ell) \cup (j_0 < c \leq j)} h_{a,b,c} A'_{a,b,c} \\
&\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{m_{i-1}n_{\ell-1}k_{j-1}} + \left(\sup_{a \geq i_0 \cup b \geq \ell_0 \cup c \geq j_0} A'_{a,b,c} \right) \frac{1}{m_{i-1}n_{\ell-1}k_{j-1}} \\
&\quad \sum_{(i_0 < a \leq i) \cup (\ell_0 < b \leq \ell) \cup (j_0 < c \leq j)} h_{a,b,c} \\
&\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{m_{i-1}n_{\ell-1}k_{j-1}} + \frac{\epsilon}{m_{i-1}n_{\ell-1}k_{j-1}} \sum_{(i_0 < a \leq i) \cup (\ell_0 < b \leq \ell) \cup (j_0 < c \leq j)} h_{a,b,c} \\
&\leq \frac{G' m_{i_0} n_{\ell_0} k_{j_0} i_0 \ell_0 j_0}{m_{i-1}n_{\ell-1}k_{j-1}} + \epsilon H^3.
\end{aligned}$$

Since m_i , n_ℓ and k_j both approaches infinity as both p, q and t approaches infinity, it follows that

$$\begin{aligned}
&\frac{1}{pqt} \sum_{m=1}^p \sum_{n=1}^q \sum_{k=1}^t \left[(|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} \right]^{p_{mnk}} = \\
&\quad 0, \text{ uniformly in } i, \ell \text{ and } j.
\end{aligned}$$

Hence $x \in \Gamma_f^3(P)$.

3.5 Theorem

Let $\theta_{i,\ell,j}$ be a triple lacunary sequence then

- (i) $(x_{mnk}) \xrightarrow{P} \Gamma^3(\widehat{S_{\theta_{i,\ell,j}}})$
- (ii) $(AC_{\theta_{i,\ell,j}})$ is a proper subset of $(\widehat{S_{\theta_{i,\ell,j}}})$
- (iii) If $x \in \Lambda^3$ and $(x_{mnk}) \xrightarrow{P} \Gamma^3(\widehat{S_{\theta_{i,\ell,j}}})$ then $(x_{mnk}) \xrightarrow{P} \Gamma^3(AC_{\theta_{i,\ell,j}})$
- (iv) $\Gamma^3(\widehat{S_{\theta_{i,\ell,j}}}) \cap \Lambda^3 = \Gamma^3[AC_{\theta_{i,\ell,j}}] \cap \Lambda^3$.

Proof: (i) Since for all i, ℓ and j

$$\begin{aligned}
&\left| \left\{ (m, n, k) \in I_{i,\ell,j} : (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} \right\} = 0 \right| \leq \\
&\sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j} \text{ and } |x_{m+i,n+\ell,k+j}|=0} (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} \leq \\
&\sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k}, \text{ for all } i, \ell \text{ and } j
\end{aligned}$$

$$P\text{-}\lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0$$

This implies that for all i, ℓ and j

$$\lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left| \left\{ (m, n, k) \in I_{i,\ell,j} : (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0 \right\} \right| = 0.$$

(ii) let $x = (x_{mnk})$ be defined as follows:

$$(x_{mnk}) = \begin{bmatrix} 1 & 2 & 3 & \dots & \frac{[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}}{(1)!} & 0 & \dots \\ 1 & 2 & 3 & \dots & \frac{[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}}{(1)!} & 0 & \dots \\ \vdots & & & & & & \\ 1 & 2 & 3 & \dots & \frac{[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}}{(1)!} & 0 & \dots \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \end{bmatrix};$$

Here x is a trible sequence and for all i, ℓ and j

$$P\text{-}\lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left| \left\{ (m, n, k) \in I_{i,\ell,j} : (|x_{m+i,n+\ell,k+j} - 0|)^{1/m+n+k} = 0 \right\} \right| =$$

$$P\text{-}\lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left(\frac{[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}}{(1)!} \right)^{1/m+n+k} = 0.$$

Therefore $(x_{mnk}) \xrightarrow{P} \Gamma^3(\widehat{S_{\theta_{i,\ell,j}}})$. Also

$$P\text{-}\lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} (|x_{m+i,n+\ell,k+j}|)^{1/m+n+k} =$$

$$P\text{-}\frac{1}{2} \left(\lim_{i,\ell,j} \frac{1}{h_{i,\ell,j}} \left(\frac{[\sqrt[4]{h_{i,\ell,j}}]^{m+n+k} [\sqrt[4]{h_{i,\ell,j}}]^{m+n+k} [\sqrt[4]{h_{i,\ell,j}}]^{m+n+k}}{(1)!} \right)^{1/m+n+k} + 1 \right) =$$

$$\frac{1}{2}.$$

Therefore $(x_{mnk}) \not\xrightarrow{P} \Gamma^3(AC_{\theta_{i,\ell,j}})$.

(iii) If $x \in \Lambda^3$ and $(x_{mnk}) \xrightarrow{P} \Gamma^3(\widehat{S_{\theta_{i,\ell,j}}})$ then $(x_{mnk}) \xrightarrow{P} \Gamma^3(AC_{\theta_{i,\ell,j}})$.

Suppose $x \in \Lambda^3$ then for all i, ℓ and j , $(|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k} \leq M$ for all m, n, k . Also for given $\epsilon > 0$ and i, ℓ and j large for all i, ℓ and j we obtain the following:

$$\begin{aligned} & \frac{1}{h_{i\ell j}} \sum_{m \in I_{i, \ell, j}} \sum_{n \in I_{i, \ell, j}} \sum_{k \in I_{i, \ell, j}} (|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k} = \\ & \frac{1}{h_{i\ell j}} \sum_{m \in I_{k, \ell}} \sum_{n \in I_{i, \ell, j}} \sum_{k \in I_{k, \ell, j} \text{ and } |x_{m+i, n+\ell, k+j}| \geq 0} (|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k} + \\ & \frac{1}{h_{i\ell j}} \sum_{m \in I_{i, \ell, j}} \sum_{n \in I_{i, \ell, j}} \sum_{k \in I_{i, \ell, j} \text{ and } |x_{m+i, n+\ell, k+j}| \leq 0} (|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k} \\ & \leq \frac{M}{h_{i\ell j}} \left| \left\{ (m, n, k) \in I_{i, \ell, j} : (|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k} \right\} = 0 \right| + \epsilon. \end{aligned}$$

Therefore $x \in \Lambda^3$ and $(x_{mnk}) \xrightarrow{P} \Gamma^3(\widehat{S_{\theta_{i, \ell, j}}})$ then $(x_{mnk}) \xrightarrow{P} \Gamma^3(AC_{\theta_{i, \ell, j}})$.

(iv) $\Gamma^3(\widehat{S_{\theta_{i, \ell, j}}}) \cap \Lambda^3 = \Gamma^3[AC_{\theta_{i, \ell, j}}] \cap \Lambda^3$. follows from (i), (ii) and (iii).

3.6 Theorem

If f be any Orlicz function then $\Gamma_f^3[AC_{\theta_{i, \ell, j}}] \notin \Gamma^3(\widehat{S_{\theta_{i, \ell, j}}})$

Proof: Let $x \in \Gamma_f^3[AC_{\theta_{i, \ell, j}}]$, for all i, ℓ and j .

Therefore we have

$$\begin{aligned} & \frac{1}{h_{i\ell j}} \sum_{m \in I_{i, \ell, j}} \sum_{n \in I_{i, \ell, j}} \sum_{k \in I_{i, \ell, j}} f \left[(|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k} \right] \geq \\ & \frac{1}{h_{i\ell j}} \sum_{m \in I_{i, \ell, j}} \sum_{n \in I_{i, \ell, j}} \sum_{k \in I_{i, \ell, j} \text{ and } |x_{m+r, n+s, k+u}| = 0} f \left[(|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k} \right] > \\ & \frac{1}{h_{i\ell j}} f(0) \left| \left\{ (m, n, k) \in I_{i, \ell, j} : (|x_{m+i, n+\ell, k+j} - 0|)^{1/m+n+k} \right\} = 0 \right|. \end{aligned}$$

Hence $x \notin \Gamma^3(\widehat{S_{\theta_{i, \ell, j}}})$.

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References

- [1] A. Esi, On some triple almost lacunary sequence spaces defined by Orlicz functions, *Research and Reviews: Discrete Mathematical Structures*, **1(2)**, (2014), 16-25.

- [2] A. Esi and M. Necdet Catalbas, Almost convergence of triple sequences, *Global Journal of Mathematical Analysis*, **2(1)**, (2014), 6-10.
- [3] A. Esi and E. Savas, On lacunary statistically convergent triple sequences in probabilistic normed space, *Appl. Math. and Inf. Sci.*, **9 (5)**, (2015), 2529-2534.
- [4] A. J. Datta A. Esi and B.C. Tripathy, Statistically convergent triple sequence spaces defined by Orlicz function, *Journal of Mathematical Analysis*, **4(2)**, (2013), 16-22.
- [5] S. Debnath, B. Sarma and B.C. Das, Some generalized triple sequence spaces of real numbers, *Journal of nonlinear analysis and optimization*, **Vol. 6, No. 1** (2015), 71-79.
- [6] P.K. Kamthan and M. Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, *65 Marcel Dekker, Inc. New York*, 1981.
- [7] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.
- [8] A. Sahiner, M. Gurdal and F.K. Duden, Triple sequences and their statistical convergence, *Selcuk J. Appl. Math.*, **8 No. (2)**(2007), 49-55.
- [9] A. Sahiner, B.C. Tripathy, Some I related properties of triple sequences, *Selcuk J. Appl. Math.*, **9 No. (2)**(2008), 9-18.
- [10] N. Subramanian and A. Esi, The generalized tripled difference of χ^3 sequence spaces, *Global Journal of Mathematical Analysis*, **3 (2)** (2015), 54-60.

