

An M/G/1 retrial queueing system with atmost J number of working vacataions

M. Sundararaman, P. Rajadurai* and D. Narasimhan
Department of Mathematics, Srinivasa Ramanujan Centre,
SASTRA University, Kumbakonam,
Thanjavur, Tamilnadu, India.
Email: psdurai17@gmail.com*

Abstract

This paper deals with a single server retrial queueing system with atmost J number of working vacations and vacation interruption. The steady state probability generating functions for the orbit size and system size is obtained by using probability generating function technique. Some system performance measures are discussed. Finally, some numerical examples are presented.

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1 Introduction

In queueing theory, vacation queues and retrial queues have been intensive research topics for long time; we can find general models in vacation queues from Ke et al.[1] and in retrial queues from Artalejo and Corral [3]. For recent studies, some of the authors have developed models in the concept of working vacataion queueing models ([2], [4], [5], [6]).

In this paper, we have developed finite number of working vacations queueing models in retrial queues. The mathematical results and theory of queues of this model provide to serve a specific and convincing application in the telecommunication system and telephone consultation of medical service systems.

2 Description of the model

In this section, we consider a single server retrial queueing system with modified working vacations and vacation interruption. The detailed description of model is given as follows:

The arrival process: Customers arrive at the system according to a Poisson process with rate λ .

The retrial process: We assume that there is no waiting space and therefore if an arriving customer finds the server free, the customer begins his service immediately, otherwise an arriving customer finds the server busy or on working vacation, the arrivals join pool of blocked customers called an orbit in accordance with FCFS discipline. That is, only one customer at the head of the orbit queue is allowed access to the server. Inter-retrial times have an arbitrary distribution $R(t)$ with corresponding Laplace Stieltjes Transform (LST) $R^*(\vartheta)$.

The working vacation process: The server begins a working vacation each time when the orbit becomes empty and the vacation time follows an exponential distribution with parameter θ . If any customer arrives in a vacation period, the server continues to work at a lower speed service rate ($\mu_v < \mu_b$). The working vacation period is an operational period at a lower speed. If any customers in the system at service completion instant in the vacation period, the server will stop the vacation and come back to the normal busy state immediately which means vacation interruption happens. Otherwise, if no customers are in the system at the end of the vacation or at regular service completion instant, the server takes at most J number working vacations until at least one customer is received in the orbit when the server returns from a working vacation. After completion of J^{th} working vacation, if there is no customer in the orbit, the server remains idle to serve a new customer. When a vacation ends, if there are customers in the orbit, the server switches to the normal working level. During the working vacation period, the service time follows a general random variable S_v with distribution function $S_v(t)$ having LST $S_v^*(\vartheta)$ for $(i = 1, 2, \dots, J)$.

The regular service process: Whenever a new customer or retry customer arrives at the server idle state, the server immediately starts normal service for the arrivals. The service time fol-

lows a general distribution. It is denoted by the random variable S_b with distribution function $S_b(t)$ having LST $S_b^*(\vartheta)$. We assume that inter-arrival times, retrial times, service times and working vacation times are mutually independent of each other.

In steady state, we assume that $R(0) = 0$, $R(\infty) = 1$, $S_b(0) = 0$, $S_b(\infty) = 1$, $S_v(0) = 0$, $S_v(\infty) = 1$ are continuous at $x = 0$. So that the function $a(x)$, $\mu_b(x)$ and $\mu_v(x)$ are the conditional completion rates for retrial, normal service and lower rate service respectively, i.e. $a(x)dx = \frac{dR(x)}{1-R(x)}$; $\mu_b(x)dx = \frac{dS_b(x)}{1-S_b(x)}$; $\mu_v(x)dx = \frac{dS_v(x)}{1-S_v(x)}$;

In addition, let $R^0(t)$, $S_b^0(t)$ and $S_v^0(t)$ be the elapsed retrial time, elapsed normal service time and elapsed working vacation time respectively at time t . Further, we introduce the random variable, for $(i = 1, 2, \dots, J)$ where, $C(t)=0$, if the server is free and in i^{th} working vacation period. $C(t)=1$, if the server is free and in regular service period. $C(t)=2$, if the server is busy and in regular service period at time t . $C(t)=3$, if the server is busy and in i^{th} working vacation period at time t .

Let $\{t_n; n = 1, 2, \dots\}$ be the sequence of epochs at which either a normal service period or working vacation period ends. The sequence of random vectors $Z_n = \{C(t_n+), N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system. The embedded Markov chain $\{Z_n; n \in N\}$ is ergodic if and only if $\lambda\beta_{(1)} < R^*(\lambda)$.

3 Steady state analysis

In this section, we develop the steady state difference-differential equations for the retrial queueing system then we derive the probability generating function (PGF) for the server states, the PGF for number of customers in the system and orbit.

For the process $\{N(t), t \geq 0\}$, we define the probabilities for $t \geq 0$, $(x, y) \geq 0$ and $n \geq 0$ and $Q_{i,0}(t) = P\{C(t) = 0, N(t) = 0\}$; $P_0(t) = P\{C(t) = 1, N(t) = 0\}$ the probability densities for $(i = 1, 2, \dots, J)$, $P_n(x, t)dx = P\{C(t) = 1, N(t) = n, x \leq R^0(t) < x + dx\}$ $\Pi_{b,n}(x, t)dx = P\{C(t) = 2, N(t) = n, x \leq S_b^0(t) < x + dx\}$ $Q_{i,n}(x, t)dx = P\{C(t) = 3, N(t) = n, x \leq S_v^0(t) < x + dx\}$.

We assume that the stability condition is fulfilled in the sequel and so that we can set $Q_{i,0} = \lim_{t \rightarrow \infty} Q_{i,0}(t)$; $P_0 = \lim_{t \rightarrow \infty} P_0(t)$ and $\lim_{t \rightarrow \infty} P_n(x, t) = P_n(x)$.

iting densities for $t \geq 0, x \geq 0$ and $n \geq 1$, $P_n(x) = \lim_{t \rightarrow \infty} P_n(x, t)$; $\Pi_{b,n}(x) = \lim_{t \rightarrow \infty} \Pi_{b,n}(x, t)$ and $\Pi_{i,n}(x) = \lim_{t \rightarrow \infty} \Pi_{i,n}(x, t)$.

By the method of supplementary variable technique, we obtain the following system of equations,

$$\lambda P_0 = \theta Q_{J,0} \quad (1)$$

$$(\lambda + \theta)Q_{1,0} = \int_{\infty}^0 \Pi_{b,0}(x)\mu_b(x)dx + \int_{\infty}^0 \Pi_{1,0}(x)\mu_v(x)dx \quad (2)$$

$$(\lambda + \theta)Q_{i,0} = \theta Q_{i-1,0} + \int_{\infty}^0 \Pi_{i,0}(x)\mu_v(x)dx, (i = 2, 3, \dots, J) \quad (3)$$

$$\frac{dP_n(x)}{dx} + (\lambda + a(x))P_n(x) = 0, n \geq 1 \quad (4)$$

$$\frac{d\Pi_{b,n}(x)}{dx} + (\lambda + \mu_b(x))\Pi_{b,n}(x) = \lambda \Pi_{b,n-1}(x), n \geq 1 \quad (5)$$

$$\frac{d\Pi_{i,n}(x)}{dx} + (\lambda + \theta + \mu_v(x))\Pi_{i,n}(x) = \lambda \Pi_{i,n-1}(x), n \geq 1 \quad (6)$$

The steady state boundary conditions at $x = 0$ and $y = 0$ are

$$P_n(0) = \int_{\infty}^0 \Pi_{b,n}(x)\mu_b(x)dx + \sum_{i=1}^J \int_{\infty}^0 \Pi_{i,n}(x)\mu_v(x)dx, n \geq 1 \quad (7)$$

$$\Pi_{b,0}(0) = \int_{\infty}^0 P_1(x)a(x)dx + \theta \int_{\infty}^0 \Pi_{i,0}(x)dx + \lambda P_0, n = 0 \quad (8)$$

$$\begin{aligned} \Pi_{b,n}(0) = & \int_{\infty}^0 P_{n+1}(x)a(x)dx + \lambda \int_{\infty}^0 P_n(x)dx \\ & + \theta \sum_{i=1}^J \int_{\infty}^0 \Pi_{i,n}(x)dx, n \geq 1 \end{aligned} \quad (9)$$

$$\Pi_{i,n}(0) = \lambda \Pi_{i,0}, n = 0, \text{ for } (i = 1, 2, \dots, J) \quad (10)$$

The normalizing condition is

$$\begin{aligned} P_0 + \sum_{i=1}^J Q_{i,0} + \sum_{n=1}^{\infty} \int_{\infty}^0 P_n(x)dx \\ + \sum_{n=0}^{\infty} \left(\int_{\infty}^0 \Pi_{b,n}(x)dx + \sum_{i=1}^J \int_{\infty}^0 \Pi_{i,n}(x)dx \right) = 1 \end{aligned} \quad (11)$$

3.1 The steady state solution

The probability generating function technique is used here to obtain the steady state solution of the retrial queueing model. We define the generating functions for $|z| = 1$ as follows: $P(x, z) = \sum_{n=1}^{\infty} P_n(x)z^n$; $\Pi_b(x, z) = \sum_{n=0}^{\infty} \Pi_{b,n}(x)z^n$; $\Pi_i(x, z) = \sum_{n=0}^{\infty} \Pi_{i,n}(x)z^n$. Now multiplying the steady state equation and steady state boundary condition (4) - (11) by z_n and summing over n , ($n = 0, 1, 2, \dots$) and solving the partial differential equations, we get the limiting PGFs $P(x, z)$, $\Pi_b(x, z)$ and $\Pi_i(x, z)$. Next we are interested in investigating the marginal orbit size distributions due to system state of the server.

Theorem 1. Under the stability condition $\lambda\beta_{(1)} < R^*(\lambda)$ the joint probability distributions of the number of customers in the orbit when server being idle, busy and working vacation is given by

$$P(z) = \frac{Nr(z)}{Dr(z)} \quad (12)$$

$$\text{where } Nr(z) = z(1 - R^*(\lambda))(P_0(S_b^*(A_b(z)) - 1)$$

$$+ \sum_{i=1}^J Q_{i,0} [V(z)S_b^*(A_b(z)) + S_b^*(A_b(z)) - 1]$$

$$Dr(z) = z - [R^*(\lambda) + z(1 - R^*(\lambda))](S_b^*(A_b(z)))$$

$$\Pi_b(z) = \lambda [1 - S_b^*(A_b(z))]$$

$$\times \frac{(P_0 R^*(\lambda)(z - 1) + \sum_{i=1}^J Q_{i,0} [zV(z) + S_v^*(A_v(z)) - 1])}{(A_b(z)) \times Dr(z)} \quad (13)$$

$$\Pi_i(z) = \frac{\lambda V(z)Q_{i,0}}{\theta}, \text{ for } (i = 1, 2, \dots, J) \quad (14)$$

$$\text{where } P_0 R^*(\lambda) + \sum_{i=1}^J Q_{i,0} [R^*(\lambda) + (\lambda/\theta)(1 - S_v^*(A_v(z)))] \\ + \lambda\beta_{(1)} ((1 - S_v^*(A_v(z)))(1 - R^*(\lambda))] = R^*(\lambda) - \lambda\beta_{(1)} \quad (15)$$

Proof. Integrating the above partial generating functions w.r.to x , then define the partial PGFs as $P(z) = \int_{-\infty}^0 P(x, z)dx$, $\Pi_b(z) = \int_{-\infty}^0 \Pi_b(x, z)dx$, $\Pi_i(z) = \int_{-\infty}^0 \Pi_i(x, z)dx$. By using the normalizing condition, the probability that the server is idle when no customer in the orbit (P_0) and ($Q_{i,0}$) can be determined by setting $z = 1$ in

the above equations ($P_0 + \sum_{i=1}^J Q_{i,0} + P(1) + \Pi_b(1) + \sum_{i=1}^J \Pi_i(1) = 1$) and applying L-Hospital's rule whenever necessary. $\square$

Corollary : Under the stability condition $\rho < R^*(\lambda)$,

(i) The PGF function of number of customers in the system

$$K_s(z) = P_0 + \sum_{i=1}^J Q_{i,0} + P(z) + z(\Pi_b(z) + \sum_{i=1}^J \Pi_i(z)). \quad (16)$$

(ii) The PGF function of number of customers in the orbit

$$K_o(z) = P_0 + \sum_{i=1}^J Q_{i,0} + P(z) + \Pi_b(z) + \sum_{i=1}^J \Pi_i(z). \quad (17)$$

4 Performance Measures

In this section, we derive some system performance measures of the model.

(i) The expected number of customers in the orbit (L_q) is obtained by differentiating (17) w.r.to z and evaluating at $z = 1$

$$L_q = K'_o(1) = \lim_{z \rightarrow 1} \frac{d}{dz} K_o(z) = \frac{Nr_q''' Dr_q'' - Nr_q'' Dr_q'''}{3(Dr_q'')^2} \quad (18)$$

$$\begin{aligned} Nr_q'' &= 2P_0 R^*(\lambda) + \sum_{i=1}^J Q_{i,0} ((1 + (\lambda(1 - S_v^*(\theta))))/\theta) (Dr_q'') \\ &\quad - 2\lambda\beta_{(1)}(1 - S_v^*(\theta))(2 - R^*(\lambda) + (\lambda/\theta)) \\ &\quad - 2(\lambda/\theta)(1 - R^*(\lambda))(1 - S_v^*(\theta)) \\ Nr_q''' &= -3 \sum_{i=1}^J Q_{i,0} (\tau((1 - S_v^*(\theta))(2 - R^*(\lambda)) - 1) \\ &\quad + 2\lambda\beta_{(1)}[V'(1) + (1 - R^*(\lambda))(\lambda S_v^{*'}(\theta) - S_v^*(\theta))] \\ &\quad + 2(\lambda/\theta)[V'(1) + (1 - R^*(\lambda))(1 - S_v^*(\theta))]) \\ Dr_q'' &= -2(R^*(\lambda) - \lambda\beta_{(1)}) \\ Dr_q''' &= 3(\lambda^2(\beta_{(2)} + 2(1 - R^*(\lambda))\lambda\beta_{(1)})) \end{aligned}$$

where $V'(1) = \frac{\lambda}{\theta}(1 - (S_v^*(\theta)) + \theta(S_v^{*'}(\theta)))$; $S_v^{*'}(\theta) = \int_0^\infty x e^{-\theta x} dS_v(x)$;

(ii) The expected number of customers in the system (L_s) is obtained by differentiating (16) w.r.to z and evaluating at $z = 1$

$$L_s = K'_s(1) = \lim_{z \rightarrow 1} \frac{d}{dz} K_s(z) = \frac{Nr_s''' Dr_q'' - Nr_q'' Dr_q'''}{3(Dr_q'')^2} \quad (19)$$

$$Nr_s''' = Nr_q''' + 6\lambda\beta_{(1)}[P_0 R^*(\lambda) - \sum_{i=1}^J Q_{i,0}(1 - S_v^*(\theta))(1 + \frac{\lambda}{\theta})].$$

(iii) The average time a customer spends in the system (W_s) and the average time a customer spends in the queue (W_q) are found by using the Little's formula $W_s = \frac{L_s}{\lambda}$ and $W_q = \frac{L_q}{\lambda}$.

5 Numerical examples

In this section, we present some numerical examples using MATLAB in order to illustrate the effect of various parameters in the system performance measures. The arbitrary values to the parameters are so chosen such that they satisfy the stability condition.

For the effect of the parameters a , θ , μ_v and μ_b on the system performance measures and the graphs are illustrated in Figure 1- Figure 2. In figure 1, we examine the behavior of the probability that server is idle (P_0) increases for increasing the value of the lower service rate (μ_v) and retrial rate (a). In figure 2, we examine the behavior of the mean orbit size (L_q) decreases for increasing the value of vacation rate (θ) and retrial rate (a).

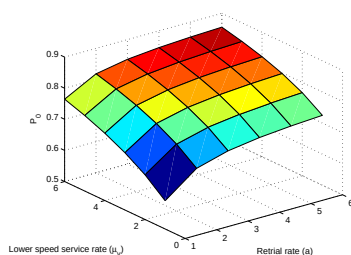


Figure 1: P_0 versus a and μ_v

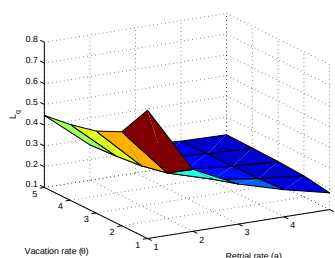


Figure 2: L_q versus θ and a

From the above numerical examples, we can find the influence of parameters on the performance measures in the system and know that the results are coincident with the practical situations.

6 Conclusion

In this paper, we have investigated an $M/G/1$ retrial queueing system with atmost J number of working vacations and vacation interruption. The probability generating functions for the numbers of customers in the system and its orbit when it is free, busy and on working vacation are found by using the supplementary variable technique. Some important System performance measures like mean system size and orbit size are obtained. Finally, numerical examples are presented. Motivation of this model is the introduction of both single working vacation ($J = 1$) and multiple working vacations ($J = \infty$) in presence of retrial queue. This model finds practical application in telephone consultation of medical service systems.

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