

A New Coupled Method for a Class of Wave Equations by the Natural Decomposition Method

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Abstract

In this paper, we have developed a new Natural Decomposition Method (NDM) for a class of wave equations with non-local boundary conditions arising in engineering. The proposed method is used to obtain the analytical solutions for a class of wave equations. The NDM is based on the Natural transform method (NTM) and the Adomian decomposition method (ADM). The obtained results are compared with ADM results. Satisfactory agreement with ADM and exact solutions is observed. Moreover, the use of NDM is found to be simple, efficient and powerful mathematical tool for a wide class of linear and nonlinear partial differential equations arising in different fields of science and engineering.

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1 Introduction

In recent years, mathematical models related to Partial Differential Equations (PDEs) are very important in science and engineering

[10]. Because theoretical validation is an essential part of research. Many theoretical models have been developed for the real time application. Important models are heat conduction models, Chemical engineering model, thermo-elasticity model and plasma physics model. Boundary value problems (BVPs) have been used to construct the solutions for the experimental validation. BVPs together with integral conditions constitute an important role in engineering. In general, the theoretical investigation or mathematical modelling is used to analyze the behaviour of the physical systems.

Recently, the analytical solutions of integral boundary problems have been established by many authors [4]. Many analytical techniques have been successfully applied for the nonlinear PDEs. But it is very difficult to get the analytical solutions for all types of nonlinear differential equations arising in engineering. Therefore semi-analytical techniques have been used to investigate the solution of nonlinear differential equations. In recent years, semi-analytical methods have been used for the nonlinear and fractional order differential equations. Some important methods are Variational iteration method (VIM) [11, 12], Adomian decomposition method (ADM) [8], homotopy perturbation method (HPM) [3] and homotopy analysis method (HAM) [9].

In this research work, we have successfully established an efficient hybrid analytical method which combines Adomian decomposition method and natural transformation method (NTM) [5, 6] for solving wave type equations. Among the applications of PDEs, the linear and nonlinear wave-type models are important study in engineering. The NTM has been successfully applied for solving the nonlinear wave-type equations. This proposed tool provides greater flexibility for choosing a proper set of basis functions for the solution. To the best of our knowledge until now there is no rigorous NTM solutions has been reported in literature. It may be concluded that the proposed method is simple, efficient and straight forward in solving both linear and nonlinear PDEs arising Engineering.

2 Wave-type equations arising in engineering

Assume that the string is undergoing small amplitude transverse vibrations so that $y(x, t)$ obeys the wave equation [9]

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + q(x, t) \quad \text{for all } 0 < x < l \text{ and } 0 < t < T \quad (1)$$

Subject to the initial condition:

$$\begin{aligned} y(x, 0) &= r(x) & \text{for all } 0 < x < l \\ y_t(x, 0) &= s(x) & \text{for all } 0 < x < l \end{aligned}$$

and the non-local boundary condition:

$$\begin{aligned} y(0, t) &= p(t) & \text{for all } 0 < t < T \\ \int_0^l y(x, t) dt &= q(t) & \text{for all } 0 < t \leq l \end{aligned}$$

where r, s, p and q are known functions, we suppose that y is sufficiently smooth to produce a smooth classical solution.

3 Basic idea of the Natural Transform Method

In this section, we discuss the preliminaries of the Natural Transform Method (NTM). Assume that a function $f(t), t \in (-\infty, \infty)$ and then the general integral transform are defined as follows [1, 2, 7]:

$$\mathfrak{N}[f(t)](s) = \int_{-\infty}^{\infty} K(s, t) f(t) dt \quad (2)$$

where $K(s, t)$ gives the kernel of the transform, s is the real (complex) number. It is independent of t . Also Note that when $K(s, t)$ is e^{-st} , $tJ_n(st)$ and $t^{s-t}(st)$, then Eq.(2) gives Laplace transform, Hankel transform and Mellin transform respectively.

Now, for $f(t), t \in (-\infty, \infty)$ consider the integral transforms defined by:

$$\mathfrak{S}[f(t)](u) = \int_{-\infty}^{\infty} K(t) f(ut) dt \quad (3)$$

and

$$\mathfrak{S}[f(t)](s, u) = \int_{-\infty}^{\infty} K(s, t) f(ut) dt \quad (4)$$

When $K(t) = e^{-t}$, Eq.(3) represents the integral Sumudu transform, where the parameter s replaced by u . Moreover, for any value of n the generalized Laplace and Sumudu transform are respectively defined by [1, 2, 7]:

$$L[f(t)] = F(s) = s^n \int_0^{\infty} e^{-s^{n+1}t} f(s^n t) dt \quad (5)$$

and

$$S[f(t)] = G(s) = u^n \int_0^{\infty} e^{-u^n t} f(tu^{n+1}) dt \quad (6)$$

When $n = 0$, Eq.(5) and Eq.(6) are the Laplace and Sumudu transform, respectively.

4 Definitions and properties of the N-Transform

The natural transform of the function $f(t)$ for $t \in (-\infty, \infty)$ is defined by [1, 2]:

$$N^+[f(t)] = R(s, u) = u^n \int_{-\infty}^{\infty} e^{-st} f(ut) dt; \quad s, u \in (-\infty, \infty) \quad (7)$$

where $N^+[f(t)]$ is the natural transformation of the time function $f(t)$ and the variables s and u are the natural transform variables. Note that Eq.(7) can be written in the form [7, 9]:

$$\begin{aligned}
N^+[f(t)] &= \int_{-\infty}^{\infty} e^{-st} f(ut) dt; \quad s, u \in (-\infty, \infty) \\
&= \left[\int_{-\infty}^0 e^{-st} f(ut) dt; \quad s, u \in (-\infty, 0) \right] \\
&\quad + \left[\int_0^{\infty} e^{-st} f(ut) dt; \quad s, u \in (0, \infty) \right] \\
&= N^-[f(t)] + N^+[f(t)] \\
&= N^-[f(t)H(-t)] + N^+[f(t)H(t)] \\
&= R^-(s, u) + R^+(s, u)
\end{aligned}$$

where $H(\cdot)$ is the Heaviside function.

We notice that if the function $f(t)H(t)$ is defined on the positive real axis, with $t \in R$, then we define the Natural transform (N -Transform) on the set

$$A = f(t) : \exists M, \tau_1, \tau_2 > 0,$$

$$\text{such that } |f(t)| < Me^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty), j \in Z^+$$

as:

$$N^+[f(t)H(t)] = N^+[f(t)] = R^+(s, u) = \int_0^{\infty} e^{-st} f(ut) dt; \quad s, u \in (0, \infty)$$

where $H(\cdot)$ is the Heaviside function. Note if $u = 1$, then Eq.(8) can be reduced to the Laplace transform and if $s = 1$, then Eq.(8) reduces to the Sumudu transform. Now we give some of the N -transforms and the conversion to Sumudu and Laplace [1, 2].

We can write the natural transform of partial derivatives in the same way [2] as following:

$$\begin{aligned}
N^+ \left[\frac{\partial y(x, t)}{\partial t} \right] &= \frac{1}{u} [sN^+[y(x, t)] - y(x, 0)] \\
N^+ \left[\frac{\partial^2 y(x, t)}{\partial t^2} \right] &= \frac{1}{u^2} [s^2 N^+[y(x, t)] - sy(x, 0) - yy'(x, t) |_{t=0}] \\
&\vdots \\
N^+ \left[\frac{\partial^n y(x, t)}{\partial t^n} \right] &= \frac{1}{u^n} \left[s^n N^+[y(x, t)] - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}} y^k(x, t) |_{t=0} \right] \quad (8)
\end{aligned}$$

Table 1: **Special N-Transforms and the conversion to Sumudu and Laplace**

$f(t)$	$N^+[f(t)]$
1	$\frac{1}{s}$
t	$\frac{u}{s^2}$
e^{at}	$\frac{1}{s-au}$
$\frac{t^{n-1}}{(n-1)!}, n = 1, 2, \dots$	$\frac{u^{n-1}}{s^n}$
$\sin(t)$	$\frac{u}{s^2+u^2}$

5 The Natural Transform Decomposition Method

The Natural Decomposition Method (NDM) algorithm by considering the general nonlinear non-homogeneous PDEs of the form:

$$Ly(x, t) + Ry(x, t) + Ny(x, t) = h(x, t) \quad (9)$$

with subject to initial condition

$$y(x, 0) = f(x); \quad y_t(x, 0) = g(x) \quad (10)$$

Here L represents the second order linear differential operator for $L = \frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order L , N is the general nonlinear differential operator and $h(x, t)$ is the source term. We apply the N -Transform to Eq.(9) to get:

$$N^+[Ly(x, t)] + N^+[Ry(x, t)] + N^+[Ny(x, t)] = N^+[h(x, t)] \quad (11)$$

Using the Property Eq.(8) and the properties:

$$\begin{aligned} \frac{s^2}{u^2}R(x, s, u) - \frac{sy(x, 0)}{u^2} - \frac{y'(x, 0)}{u} \\ + N^+[Ry(x, t)] + N^+[Ny(x, t)] = N^+[h(x, t)] \end{aligned} \quad (12)$$

Substitute Eq.(10) into Eq.(??) to get:

$$\begin{aligned} \frac{s^2}{u^2}R(x, s, u) - \frac{sf(x)}{u^2} - \frac{g(x)}{u} + N^+[Ry(x, t)] \\ + N^+[Ny(x, t)] = N^+[h(x, t)] \end{aligned} \quad (13)$$

From Eq.(??) we get

$$\begin{aligned} \frac{s^2}{u^2}R(x, s, u) = \frac{sf(x)}{u^2} + \frac{g(x)}{u} \\ + N^+[h(x, t)] - N^+[Ry(x, t)] - N^+[Ny(x, t)] \end{aligned} \quad (14)$$

Eq.(14) becomes

$$\begin{aligned} R(x, s, u) = \frac{f(x)}{s} + \frac{ug(x)}{s^2} + \frac{u^2}{s^2}N^+[h(x, t)] \\ - \frac{u^2}{s^2}N^+[Ry(x, t)] - \frac{u^2}{s^2}N^+[Ny(x, t)] \end{aligned} \quad (15)$$

We take the inverse Natural transform of Eq.(15),

$$\begin{aligned} N^{-1}R(x, s, u) = N^{-1} \left[\frac{f(x)}{s} + \frac{ug(x)}{s^2} + \frac{u^2}{s^2}N^+[h(x, t)] \right] \\ - N^{-1} \left[\frac{u^2}{s^2}N^+[Ry(x, t)] \right] - N^{-1} \left[\frac{u^2}{s^2}N^+[Ny(x, t)] \right] \end{aligned}$$

From Eq.(15) we can get

$$y(x, t) = H(x, t) - N^{-1} \left[\frac{u^2}{s^2}N^+[Ry(x, t)] + \frac{u^2}{s^2}N^+[Ny(x, t)] \right] \quad (16)$$

Note $H(x, t)$ is the term arising from the source term and the prescribed initial conditions. Now to deal with nonlinear term, we represent the solution in an infinite series form:

$$y(x, t) = \sum_{n=0}^{\infty} y_n(x, t) \quad (17)$$

Also, the nonlinear term can be written as:

$$Ny(x, t) = \sum_{n=0}^{\infty} A_n \quad (18)$$

where the A_n s are the polynomials of $u_0, u_1, u_2, u_3, \dots, u_n$ and can be calculated by the following formula:

$$A_n = \frac{1}{n!} \frac{d^n}{dx^n} \left[F \left(\sum_{i=0}^n \lambda^i y_i \right) \right], \quad n = 0, 1, 2, \dots \quad (19)$$

The general formula in Eq.(19), can be simplified as follows: We first assume that the nonlinear function is $F(u)$, the Adomian polynomials are given by:

$$\begin{aligned} A_0 &= F(y_0) \\ A_1 &= u_1 F'(y_0) \\ A_2 &= u_2 F'(y_0) + \frac{1}{2!} y_1^2 F''(y_0) \end{aligned}$$

And so on. Now, we substitute Eq.(17) and Eq.(18) into Eq.(16) to get:

$$\sum_{n=0}^{\infty} y_n(x, t) = H(x, t) - N^{-1} \left[\frac{u^2}{s^2} N^+ \left[R \sum_{n=0}^{\infty} y_n(x, t) \right] + \sum_{n=0}^{\infty} A_n \right] \quad (20)$$

Comparing both sides of Eq.(20) we conclude

$$\begin{aligned} y_0(x, t) &= H(x, t) \\ y_1(x, t) &= -N^{-1} \left[\frac{u^2}{s^2} N^+ [Ry_0(x, t)] + A_0 \right] \\ y_2(x, t) &= -N^{-1} \left[\frac{u^2}{s^2} N^+ [Ry_1(x, t)] + A_1 \right] \end{aligned}$$

The general recursive relation is given by:

$$y_{n+1}(x, t) = -N^{-1} \left[\frac{u^2}{s^2} N^+ [Ry_n(x, t)] + A_n \right], \quad n \geq 1 \quad (21)$$

Hence, from the general recursive relation in Eq.(21) we can easily compute the remaining components of $y(x, t)$ as $y_1(x, t), y_2(x, t), \dots$ where $y_0(x, t)$ is always the given initial condition.

Finally, the exact solution is given by:

$$y(x, t) = \sum_{n=0}^{\infty} y_n(x, t) \quad (22)$$

6 Applications

In this section, we Implement the NDM to five numerical examples and then compare our solutions to existing exact solutions.

Consider the following wave equation:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < 1, \quad 0 < t < 0.5 \quad (23)$$

with the initial conditions:

$$\begin{aligned} y(x, 0) &= 0, \quad 0 \leq x \leq 1 \\ y_t(x, 0) &= \pi \cos(\pi x), \quad 0 \leq x \leq 1 \end{aligned}$$

and the boundary conditions:

$$\begin{aligned} y(0, t) &= p(t) = \sin(\pi t) \\ \int_0^t y(x, t) dt &= q(t) = 0 \end{aligned}$$

We first take the N -Transform of Eq.(23) we obtain:

$$N^+ \left[\frac{\partial^2 y}{\partial t^2} \right] = N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \quad (24)$$

Using the property Eq.(8), we get:

$$\frac{s^2}{u^2} R(x, s, u) - \frac{s}{u^2} y(x, 0) - \frac{1}{u} y'(x, 0) = N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \quad (25)$$

Using the initial conditions into Eq.(25) to get:

$$R(x, s, u) = \frac{u}{s^2} \pi \cos(\pi x) + \frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \quad (26)$$

Now we take the inverse N -Transform of Eq.(26) to get:

$$N^{-1} [R(x, s, u)] = N^{-1} \left[\frac{u}{s^2} \pi \cos(\pi x) \right] + N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (27)$$

From Table 1, Eq.(27) becomes

$$y(x, t) = N^{-1} \left[\frac{u}{s^2} \pi \cos(\pi x) \right] + N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (28)$$

Now from equation (28), we can get:

$$y(x, t) = \pi \cos(\pi x) t + N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\sum_{n=0}^{\infty} B_n \right] \right] \quad (29)$$

Now from Eq.(29) we conclude

$$\begin{aligned} y_0(x, t) &= \pi t \cos(\pi x), \\ y_1(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] \\ y_2(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [B_1] \right] \end{aligned}$$

We continue in this manner to get the general recursive relation:

$$y_{n+1}(x, t) = N^{-1} \left[\frac{u^2}{s^2} N^+ [B_n] \right], \quad n \geq 1 \quad (30)$$

Note that

$$\begin{aligned} y_1(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] = N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_0)_{xx}] \right] \\ &= N^{-1} \left[\frac{u^2}{s^2} N^+ [-\pi^3 \cos(\pi x) t] \right] = -\frac{(\pi t)^3}{3!} \cos(\pi x) \end{aligned}$$

And

$$\begin{aligned} y_2(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_1)_{xx}] \right] = N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\pi^5 t^3}{3!} \cos(\pi x) \right] \right] \\ &= \frac{(\pi t)^5}{5!} \cos(\pi x) \\ y_3(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_2)_{xx}] \right] = N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{-\pi^7 t^5}{5!} \cos(\pi x) \right] \right] \\ &= -\frac{(\pi t)^7}{7!} \cos(\pi x) \end{aligned}$$

Figure 1: **Exact Solution**
 [width=13cm,height=6cm]Examples1and2plots

We continue in this manner to get:

$$\begin{aligned} y(x, t) &= y_0(x, t) + y_1(x, t) + y_2(x, t) + y_3(x, t) + \dots \\ &= \cos(\pi x) \left[\pi t - \frac{(\pi t)^3}{3!} + \frac{(\pi t)^5}{5!} - \frac{(\pi t)^7}{7!} + \dots \right] \\ &= \cos(\pi x) \sin(\pi t) \end{aligned}$$

This converges to the exact solution.

Consider the following wave equation:

$$\frac{\partial^2 y}{\partial t^2} - \frac{1}{4} \frac{\partial^2 y}{\partial x^2} = 0, \quad 0 < x < 1, \quad 0 < t < 1 \quad (31)$$

with the initial conditions:

$$\begin{aligned} y(x, 0) &= x, \quad 0 < x < 1 \\ y_t(x, 0) &= e^x, \quad 0 < x < 1 \end{aligned}$$

and the boundary conditions:

$$\begin{aligned} y_x(0, t) &= 2 \sinh\left(\frac{t}{2}\right), \quad t > 0 \\ y_x(1, t) &= 2e^x \left(\sinh\left(\frac{t}{2}\right) + 1 \right), \quad t > 0 \end{aligned}$$

We first take the N-Transform of Eq.(31) we obtain:

$$N^+ \left[\frac{\partial^2 y}{\partial t^2} \right] - \frac{1}{4} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] = 0 \quad (32)$$

Using the property Eq.(8) and the properties mentioned above we get:

$$\frac{s^2}{u^2} R(x, s, u) - \frac{s}{u^2} y(x, 0) - \frac{1}{u} y'(x, 0) - \frac{1}{4} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] = 0 \quad (33)$$

Using the initial conditions into Eq.(33) to get:

$$R(x, s, u) = \frac{x}{s} + e^x \frac{u}{s^2} + \frac{1}{4} \frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \quad (34)$$

Now we take the inverse N -Transform of Eq.(34) to get:

$$N^{-1} [R(x, s, u)] = N^{-1} \left[\frac{x}{s} + e^x \frac{u}{s^2} \right] + \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (35)$$

From Table 1, Eq.(35) becomes

$$y(x, t) = x + te^x + \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (36)$$

Now from equation (36), we can get:

$$y(x, t) = x + te^x + \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\sum_{n=0}^{\infty} B_n \right] \right] \quad (37)$$

Now from Eq.(37) we conclude

$$\begin{aligned} y_0(x, t) &= x + te^x, \\ y_1(x, t) &= \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] \\ y_2(x, t) &= \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [B_1] \right] \end{aligned}$$

We continue in this manner to get the general recursive relation:

$$y_{n+1}(x, t) = \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [B_n] \right], \quad n \geq 1 \quad (38)$$

Note that

$$\begin{aligned} y_1(x, t) &= \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] = \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_0)_{xx}] \right] \\ &= \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [te^x] \right] = \frac{e^x t^3}{4 \cdot 3!} \end{aligned}$$

And

$$\begin{aligned} y_2(x, t) &= \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_1)_{xx}] \right] = \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{e^x t^3}{4 \cdot 3!} \right] \right] \\ &= \frac{e^x t^5}{4^2 \cdot 5!} \end{aligned}$$

$$\begin{aligned}
y_3(x, t) &= \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_2)_{xx}] \right] = \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{e^x t^5}{4^2 5!} \right] \right] \\
&= \frac{e^x t^7}{4^3 7!}
\end{aligned}$$

We continue in this manner to get:

$$\begin{aligned}
y(x, t) &= y_0(x, t) + y_1(x, t) + y_2(x, t) + y_3(x, t) + \dots \\
&= x + 2e^x \left[t/2 - \frac{(t/2)^3}{3!} + \frac{(t/2)^5}{5!} - \frac{(t/2)^7}{7!} + \dots \right] \\
&= x + 2e^x \sinh \left(\frac{t}{2} \right)
\end{aligned}$$

This converges to the exact solution.

Consider the wave equation:

$$\frac{\partial^2 y}{\partial t^2} - 4 \frac{\partial^2 y}{\partial x^2} = 0, \quad 0 < x < 1, \quad 0 < t < 1 \quad (39)$$

with the initial conditions:

$$\begin{aligned}
y(x, 0) &= \sin(\pi x), \quad 0 \leq x \leq 1 \\
y_t(x, 0) &= 0, \quad 0 \leq x \leq 1
\end{aligned}$$

and the boundary conditions:

$$y(0, t) = y(1, t) = 0, \quad t > 0$$

Compare the result with the exact solution

$$y(x, t) = \sin(\pi x) \cos(2\pi t)$$

We first take the N-Transform of Eq.(39) we obtain:

$$N^+ \left[\frac{\partial^2 y}{\partial t^2} \right] - 4N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] = 0 \quad (40)$$

Using the property Eq.(8) and the properties mentioned above we get:

$$\frac{s^2}{u^2} R(x, s, u) - \frac{s}{u^2} y(x, 0) - \frac{1}{u} y'(x, 0) - 4N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] = 0 \quad (41)$$

Using the initial conditions into Eq.(41) to get:

$$R(x, s, u) = \frac{1}{s} \sin(\pi x) + 4 \frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \quad (42)$$

Now we take the inverse N -Transform of Eq.(42) to get:

$$N^{-1} [R(x, s, u)] = N^{-1} \left[\frac{1}{s} \sin(\pi x) \right] + 4 N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (43)$$

From Table 1, Eq.(43) becomes

$$y(x, t) = \sin(\pi x) + 4 N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (44)$$

Now from equation (44), we can get:

$$y(x, t) = \sin(\pi x) + 4 N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\sum_{n=0}^{\infty} B_n \right] \right] \quad (45)$$

Now from Eq.(45) we conclude

$$\begin{aligned} y_0(x, t) &= \sin(\pi x), \\ y_1(x, t) &= 4 N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] \\ y_2(x, t) &= 4 N^{-1} \left[\frac{u^2}{s^2} N^+ [B_1] \right] \end{aligned}$$

We continue in this manner to get the general recursive relation:

$$y_{n+1}(x, t) = 4 N^{-1} \left[\frac{u^2}{s^2} N^+ [B_n] \right], \quad n \geq 1 \quad (46)$$

Note that

$$\begin{aligned} y_1(x, t) &= 4 N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] = 4 N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_0)_{xx}] \right] \\ &= 4 N^{-1} \left[\frac{u^2}{s^2} N^+ [-\pi^2 \sin(\pi x)] \right] = -4 \pi^2 \sin(\pi x) \frac{t^2}{2!} \end{aligned}$$

Figure 2: **Exact Solution**

[width=12cm,height=5.5cm]Examples3and4plots

And

$$\begin{aligned}
 y_2(x, t) &= 4N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_1)_{xx}] \right] \\
 &= 4N^{-1} \left[\frac{u^2}{s^2} N^+ \left[4\pi^4 \sin(\pi x) \frac{t^2}{2!} \right] \right] \\
 &= 4^2 \pi^4 \sin(\pi x) \frac{t^4}{4!}
 \end{aligned}$$

$$\begin{aligned}
 y_3(x, t) &= \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_2)_{xx}] \right] \\
 &= \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[-4^2 \pi^6 \sin(\pi x) \frac{t^4}{4!} \right] \right] \\
 &= -4^3 \pi^6 \sin(\pi x) \frac{t^6}{6!}
 \end{aligned}$$

We continue in this manner to get:

$$\begin{aligned}
 y(x, t) &= y_0(x, t) + y_1(x, t) + y_2(x, t) + y_3(x, t) + \dots \\
 &= \sin(\pi x) \left[1 - \frac{(2\pi t)^2}{2!} + \frac{(2\pi t)^4}{4!} - \frac{(2\pi t)^6}{6!} + \dots \right] \\
 &= \sin(\pi x) \cos(2\pi t)
 \end{aligned}$$

This converges to the exact solution.

Consider the wave equation:

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0, \quad 0 < x < 1, \quad 0 < t < 1 \quad (47)$$

with the initial conditions:

$$\begin{aligned}
 y(x, 0) &= \sin(\pi x), \quad 0 \leq x \leq 1 \\
 y_t(x, 0) &= 0, \quad 0 \leq x \leq 1
 \end{aligned}$$

and the boundary conditions:

$$y(0, t) = y(1, t) = 0, \quad t > 0$$

Compare the result with the exact solution

$$y(x, t) = \sin(\pi x) \cos(2\pi t)$$

We first take the N-Transform of Eq.(47) we obtain:

$$N^+ \left[\frac{\partial^2 y}{\partial t^2} \right] - N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] = 0 \quad (48)$$

Using the property Eq.(8) and the properties mentioned above we get:

$$\frac{s^2}{u^2} R(x, s, u) - \frac{s}{u^2} y(x, 0) - \frac{1}{u} y'(x, 0) - N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] = 0 \quad (49)$$

Using the initial conditions into Eq.(49) to get:

$$R(x, s, u) = \frac{1}{s} \sin(\pi x) + \frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \quad (50)$$

Now we take the inverse N-Transform of Eq.(50) to get:

$$N^{-1} [R(x, s, u)] = N^{-1} \left[\frac{1}{s} \sin(\pi x) \right] + N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (51)$$

From Table 1, Eq.(51) becomes

$$y(x, t) = \sin(\pi x) + N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (52)$$

Now from equation (52), we can get:

$$y(x, t) = \sin(\pi x) + N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\sum_{n=0}^{\infty} B_n \right] \right] \quad (53)$$

Now from Eq.(53) we conclude

$$\begin{aligned} y_0(x, t) &= \sin(\pi x), \\ y_1(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] \\ y_2(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [B_1] \right] \end{aligned}$$

We continue in this manner to get the general recursive relation:

$$y_{n+1}(x, t) = N^{-1} \left[\frac{u^2}{s^2} N^+ [B_n] \right], \quad n \geq 1 \quad (54)$$

Note that

$$\begin{aligned} y_1(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] = N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_0)_{xx}] \right] \\ &= N^{-1} \left[\frac{u^2}{s^2} N^+ [-\pi^2 \sin(\pi x)] \right] = -\pi^2 \sin(\pi x) \frac{t^2}{2!} \end{aligned}$$

And

$$\begin{aligned} y_2(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_1)_{xx}] \right] \\ &= N^{-1} \left[\frac{u^2}{s^2} N^+ \left[4\pi^4 \sin(\pi x) \frac{t^2}{2!} \right] \right] \\ &= \pi^4 \sin(\pi x) \frac{t^4}{4!} \\ y_3(x, t) &= \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_2)_{xx}] \right] \\ &= \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[-4^2 \pi^6 \sin(\pi x) \frac{t^4}{4!} \right] \right] \\ &= -\pi^6 \sin(\pi x) \frac{t^6}{6!} \end{aligned}$$

We continue in this manner to get:

$$\begin{aligned} y(x, t) &= y_0(x, t) + y_1(x, t) + y_2(x, t) + y_3(x, t) + \dots \\ &= \sin(\pi x) \left[1 - \frac{(\pi t)^2}{2!} + \frac{(\pi t)^4}{4!} - \frac{(\pi t)^6}{6!} + \dots \right] \\ &= \sin(\pi x) \cos(\pi t) \end{aligned}$$

This converges to the exact solution.

Consider the wave equation:

$$\frac{\partial^2 y}{\partial t^2} - \frac{1}{16\pi^2} \frac{\partial^2 y}{\partial x^2} = 0, \quad 0 < x < 1, \quad 0 < t < 1 \quad (55)$$

with the initial conditions:

$$\begin{aligned} y(x, 0) &= 0, \quad 0 \leq x \leq 1 \\ y_t(x, 0) &= \sin(4\pi x), \quad 0 \leq x \leq 1 \end{aligned}$$

and the boundary conditions:

$$y(0, t) = y(0.5, t) = 0, \quad t > 0$$

Compare the result with the exact solution

$$y(x, t) = \sin(4\pi x) \sin(t)$$

We first take the N-Transform of Eq.(55) we obtain:

$$N^+ \left[\frac{\partial^2 y}{\partial t^2} \right] - \frac{1}{16\pi^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] = 0 \quad (56)$$

Using the property Eq.(8) and the properties mentioned above we get:

$$\frac{s^2}{u^2} R(x, s, u) - \frac{s}{u^2} y(x, 0) - \frac{1}{u} y'(x, 0) - \frac{1}{16\pi^2} N^+ \left[\frac{\partial^2 u}{\partial x^2} \right] = 0 \quad (57)$$

Using the initial conditions into Eq.(57) to get:

$$R(x, s, u) = \frac{u}{s^2} \sin(4\pi x) + \frac{1}{16\pi^2} \frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \quad (58)$$

Now we take the inverse N-Transform of Eq.(58) to get:

$$N^{-1} [R(x, s, u)] = N^{-1} \left[\frac{u}{s^2} \sin(4\pi x) \right] + \frac{1}{16\pi^2} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (59)$$

From Table 1, Eq.(59) becomes

$$y(x, t) = t \sin(4\pi x) + \frac{1}{16\pi^2} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (60)$$

Now from equation (60), we can get:

$$y(x, t) = t \sin(4\pi x) + \frac{1}{16\pi^2} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\sum_{n=0}^{\infty} B_n \right] \right] \quad (61)$$

Now from Eq.(61) we conclude

$$\begin{aligned} y_0(x, t) &= t \sin(4\pi x), \\ y_1(x, t) &= \frac{1}{16\pi^2} N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] \\ y_2(x, t) &= \frac{1}{16\pi^2} N^{-1} \left[\frac{u^2}{s^2} N^+ [B_1] \right] \end{aligned}$$

Figure 3: **Exact Solution for Example: 5**
 [width=10cm,height=6cm]Examples5plots

We continue in this manner to get the general recursive relation:

$$y_{n+1}(x, t) = \frac{1}{16\pi^2} N^{-1} \left[\frac{u^2}{s^2} N^+ [B_n] \right], \quad n \geq 1 \quad (62)$$

Note that

$$\begin{aligned} y_1(x, t) &= \frac{1}{16\pi^2} N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] = \frac{1}{16\pi^2} N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_0)_{xx}] \right] \\ &= \frac{1}{16\pi^2} N^{-1} \left[\frac{u^2}{s^2} N^+ [-(4\pi)^2 t \sin(4\pi x)] \right] \\ &= -\sin(4\pi x) \frac{t^3}{3!} \end{aligned}$$

And

$$\begin{aligned} y_2(x, t) &= \frac{1}{16\pi^2} N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_1)_{xx}] \right] \\ &= \frac{1}{16\pi^2} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[(4\pi)^2 \sin(4\pi x) \frac{t^3}{3!} \right] \right] \\ &= \sin(4\pi x) \frac{t^5}{5!} \\ y_3(x, t) &= \frac{1}{16\pi^2} \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_2)_{xx}] \right] \\ &= \frac{1}{16\pi^2} \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[-(4\pi)^2 \sin(4\pi x) \frac{t^5}{5!} \right] \right] \\ &= -\sin(4\pi x) \frac{t^7}{7!} \end{aligned}$$

We continue in this manner to get:

$$\begin{aligned} y(x, t) &= y_0(x, t) + y_1(x, t) + y_2(x, t) + y_3(x, t) + \dots \\ &= \sin(4\pi x) \left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots \right] = \sin(4\pi x) \sin(t) \end{aligned}$$

This converges to the exact solution.

Consider the wave equation:

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0, \quad 0 < x < \pi, \quad 0 < t \quad (63)$$

with the initial conditions:

$$\begin{aligned} y(x, 0) &= \sin(x), \quad 0 \leq x \leq \pi \\ y_t(x, 0) &= 0, \quad 0 \leq x \leq \pi \end{aligned}$$

and the boundary conditions:

$$y(0, t) = y(\pi, t) = 0, \quad t > 0$$

Compare the result with the exact solution

$$y(x, t) = \sin(x) \cos(t)$$

We first take the N-Transform of Eq.(63) we obtain:

$$N^+ \left[\frac{\partial^2 y}{\partial t^2} \right] - N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] = 0 \quad (64)$$

Using the property Eq.(8) and the properties mentioned above we get:

$$\frac{s^2}{u^2} R(x, s, u) - \frac{s}{u^2} y(x, 0) - \frac{1}{u} y'(x, 0) - N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] = 0 \quad (65)$$

Using the initial conditions into Eq.(65) to get:

$$R(x, s, u) = \frac{1}{s} \sin(x) + \frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \quad (66)$$

Now we take the inverse N-Transform of Eq.(66) to get:

$$N^{-1} [R(x, s, u)] = N^{-1} \left[\frac{1}{s} \sin(x) \right] + N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (67)$$

From Table 1, Eq.(67) becomes

$$y(x, t) = \sin(x) + N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\frac{\partial^2 y}{\partial x^2} \right] \right] \quad (68)$$

Now from equation (68), we can get:

$$y(x, t) = \sin(x) + N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\sum_{n=0}^{\infty} B_n \right] \right] \quad (69)$$

Now from Eq.(69) we conclude

$$\begin{aligned} y_0(x, t) &= \sin(x), \\ y_1(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] \\ y_2(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [B_1] \right] \end{aligned}$$

We continue in this manner to get the general recursive relation:

$$y_{n+1}(x, t) = N^{-1} \left[\frac{u^2}{s^2} N^+ [B_n] \right], \quad n \geq 1 \quad (70)$$

Note that

$$\begin{aligned} y_1(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [B_0] \right] = N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_0)_{xx}] \right] \\ &= N^{-1} \left[\frac{u^2}{s^2} N^+ [-\sin(x)] \right] = -\sin(x) \frac{t^2}{2!} \end{aligned}$$

And

$$\begin{aligned} y_2(x, t) &= N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_1)_{xx}] \right] = N^{-1} \left[\frac{u^2}{s^2} N^+ \left[\sin(x) \frac{t^2}{2!} \right] \right] \\ &= \sin(x) \frac{t^4}{4!} \\ y_3(x, t) &= \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ [(y_2)_{xx}] \right] = \frac{1}{4} N^{-1} \left[\frac{u^2}{s^2} N^+ \left[-\sin(x) \frac{t^4}{4!} \right] \right] \\ &= -\sin(x) \frac{t^6}{6!} \end{aligned}$$

We continue in this manner to get:

$$\begin{aligned} y(x, t) &= y_0(x, t) + y_1(x, t) + y_2(x, t) + y_3(x, t) + \dots \\ &= \sin(x) \left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots \right] = \sin(x) \cos(t) \end{aligned}$$

This converges to the exact solution.

7 Conclusion

In this paper, an efficient Natural Decomposition Method (NDM) has been successfully applied for the analytical solution of the wave-type equations. The obtained results have been compared with exact solutions. The accuracy of the method is also confirmed. Also the proposed method can be easily extended to solve other nonlinear partial differential equations arising in engineering. Finally, we have concluded that NDM is simple, straight forward, efficient and less computational costs.

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