

An efficient analytical/approximation method for a class of nonlinear reaction-diffusion equations arising in Engineering

M.Salai Mathi Selvi¹, G.Hariharan^{1,*} and S. Padma

¹Department of Mathematics, School of Humanities and Sciences,
SASTRA University, Thanjavur-613 401, Tamilnadu, India.

E.Mail: hariharan@maths.sastra.edu*

November 13, 2017

Abstract

In this paper, we have established the homotopy analysis method (HAM) for solving a few partial differential equations arising in engineering. This technique provides the solutions in rapid convergence series with computable terms for the problems with high degree of nonlinear terms appearing in the governing differential equations. The convergence analysis of the proposed method is also discussed. Finally, we have given some illustrative examples to demonstrate the validity and applicability of the proposed method

AMS Subject Classification: 42C05, 42C40, 34K28, 78M22, 34B181.

Keywords: Homotopy analysis method; Adomian decomposition method; Homotopy perturbation method; Haar wavelets

1 Introduction

Non-linear phenomena play a very important role in physics, chemistry and biology (heat and mass transfer, filtration of liquids, dif-

fusion in chemical reactions, etc.). Partial differential equations (PDEs) play a crucial role in applied mathematics and physics. Recently, various iterative methods are applied for getting numerical and analytical solutions of linear and non-linear partial differential equations (Abbasbandy, 2006; Borhanifar and Reza Abazari, 2010; Hariharan and Kannan, 2010; Hariharan and Rajaraman, 2013; Hariharan, 2013; Kkarslan, 2009; Hashim et al. 2006; Jafari et al. 2008; Mamontov, 2013; Molabahrani and Khani, 2009). But the nonlinear PDEs are difficult to get their exact solutions. So the approximation and numerical methods must be used. The numerical solutions of the nonlinear reaction-diffusion equations (NLRDEs) have received considerable attention in the literature and fall into two groups: The analytical methods and the numerical ones. Analytical methods enable researchers to study the effect of different variables or parameters on the function under study easily. Recently, a powerful analytic method for nonlinear problems, the so called Homotopy analysis method (HAM), has been proposed by [Liao, 2003; Liao, 2004; Liao, 2005; Liao, 2009]. Moreover, the HAM itself provides us with a convenient way to control the convergence of approximation series and adjust convergence regions when necessary. Recently, there are many new algorithms for NLRDEs have been proposed, for example, the Adomian decomposition Method [Hashim et al. 2006], the variational iteration method [Golbabai and Javidi, 2007], differential transform method [Borhanifar and Reza Abazari, 2010], homotopy perturbation method [Biazar and Ghazvini, 2007; Bizzar and Ghazvini, 2009; Domairy and Nadim, 2008; Jafari et al. 2008; Kkarslan, 2009; Kazemi-Sangsereki et al. 2011; Matinfar and Raeisi, 2011], homotopy analysis method [Molabahrani and Khani, 2009; Mamontov, 2013; Liao, 2003; Hariharan, G., 2013; Abbasbandy, S., 2006; Abbasbandy, S., 2008] and other methods [Hariharan and Rajaraman, 2013; Hariharan and Kannan, 2010; Mamontov, 2013].

2 Basic idea of homotopy analysis method (HAM) [Liao, 2003]

In this paper, we have applied the homotopy analysis method (HAM) to solve the linear and nonlinear Partial differential equations aris-

ing in mathematical chemistry. In this section the basic ideas of the homotopy analysis method are introduced. Here a description of the method [Liao,2003] is given to handle the general nonlinear problem.

$$Nu_0(t) = 0, t > 0 \quad (1)$$

Where N is a nonlinear operator and $u_0(t)$ is unknown function of the independent variable t .

2.1 Zero- order deformation equation [Liao,2005]

Let $u_0(t)$ denote the initial guess of the exact solution of Eq.(1), $h \neq 0$ an auxiliary parameter, $H(t) \neq 0$ an auxiliary function and L an auxiliary linear operator with the property.

$$L[f(t)] = 0, \quad f(t) = 0 \quad (2)$$

The auxiliary parameter h , the auxiliary function $H(t)$, and the auxiliary linear operator L play important roles within the HAM to adjust and control the convergence region of solution series.[Liao, 2005] constructs, using $q \in [0, 1]$ as an embedding parameter, the so-called zero-order deformation equation

$$(1 - q)L[\Phi(t; q) - u_0(t)] = qhH(t)N[\Phi(t; q)] \quad (3)$$

Where $\Phi(t, q)$ is the solution which depends on $h, H(t), L, u_0(t)$ and q . when $q = 0$, the zero-order deformation Eq.(2) becomes $\Phi(t; 0) = u_0(t)$ and when $q = 1$, since $h \neq 0$ and $H(t) \neq 0$, the zero-order deformation equation reduces to, $N[\Phi(t; 1)] = 0$ So, $\Phi(t; 1)$ is exactly the solution of the nonlinear Eq.(1). Define the so-called m th order deformation derivatives.

$$u_m(t) = \frac{1}{m!} \frac{\partial^m \Phi(t; q)}{\partial q^m} \quad (4)$$

If the power series Eq.(4) of $\Phi(t; q)$ converges at $q = 1$, then we gets the following series solution

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t) \quad (5)$$

where the terms $u_m(t)$ can be determined by the so-called high order deformation described below.

2.2 High- order deformation equation

Define the vector,

$$\vec{u}_n = u_0(t), u_1(t), \dots, u_n(t) \quad (6)$$

Differentiating Eq.(3) m times with respect to embedding parameter q , the setting $q = 0$ and dividing them by $m!$ we have the so-called m^{th} order deformation equation.

$$L[u_m(t) - \chi_m u_{m-1}(t)] = hH(t)\mathfrak{R}_m(\vec{u}_m, t) \quad (7)$$

where

$$\chi_m = \begin{cases} 0, & \text{for } m \leq 1 \\ 1, & \text{for } m > 1 \end{cases} \quad (8)$$

$$\mathfrak{R}_m(\vec{u}_m, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t, q)]}{\partial q^{m-1}} \quad (9)$$

For any given nonlinear operator N , the term $\mathfrak{R}_m(\vec{u}_m, t)$ can be easily expressed by Eq.(9). Thus, we can gain $u_1(t), u_2(t), \dots$ by means of solving the linear high-order deformation Eq.(7) one after the other order in order. The m^{th} order approximation of $u(t)$ is given by

$$u(t) = \sum_{k=0}^m u_k(t) \quad (10)$$

Example. 1

Consider the nonlinear in homogeneous Cauchy equation (inviscid Burger equation) arising in one dimensional stream of particles or fluid having zero viscosity. $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 2x$, $x \in R$, $t \geq 0$ with initial condition $u_0(x, t) = x$

Apply the HAM, We choose the initial approximation $u_0(x, t) = x$ We define the nonlinear operator as

$$N[\phi(x, t; q)] = \frac{\partial \phi(x, t; q)}{\partial t} + \phi(x, t; q) \frac{\partial \phi(x, t; q)}{\partial x} - 2x$$

and linear operator

$$L[\phi(x, t; q)] = \frac{\partial \phi(x, t; q)}{\partial t}$$

with the property $L(c) = 0$ where c is constant.

Now by using Eq.(7) we have

$$\mathfrak{R}_m(U_{m-1}^{\rightarrow}) = \frac{\partial \phi(x, t; q)}{\partial t} + \phi(x, t; q) \frac{\partial \phi(x, ; q)}{\partial x} - 2x$$

and the solution of the m^{th} order deformation equation Eq.(7) for $m \geq 1$ becomes

$$u_{m-1}(x, t) = \chi_m U_{m-1}(x, t) + hH(r, t)L^{-1}[\mathfrak{R}_m(U_{m-1}^{\rightarrow})]$$

since $m \geq 1$, $\chi_m = 1$, we set $h = -1$ and $H(r, t) = 1$ and we successively obtain $u_0(x, t) = x$,

$$u_1(x, t) = x + xt,$$

$$u_2(x, t) = x + xt + x\frac{t^2}{2},$$

$$u_3(x, t) = x + xt + x\frac{t^2}{2} + x\frac{t^3}{6},$$

.....

Hence

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots$$

The final solution in a closed form is given by

$$u(x, t) = xe^t$$

The surface area shows $u(x, t)$

Example. 2

Consider the homogeneous gas dynamic equation of the form $\frac{\partial u(x, t)}{\partial t} + \frac{1}{2}(\frac{\partial u^2(x, t)}{\partial x} = \frac{\partial^2 u(x, t)}{\partial x^2} - u^2(x, t)$

With initial condition: $u(x, 0) = e^{-x}$

Apply the HAM, the solution of the m^{th} order deformation equation Eq.(1) for $m \geq 1$, $\chi_m = 1$, we set $h = -1$ and $H(r, t) = 1$,

$$u_{m-1}(x, t) = \chi_m U_{m-1}(x, t) + hH(r, t)L^{-1}[\mathfrak{R}_m(U_{m-1}^{\rightarrow})]$$

where

$$\mathfrak{R}_m(U_{m-1}^{\rightarrow}) = \frac{\partial \phi(x, t; q)}{\partial t} + \frac{1}{2} \frac{\partial \phi^2(x, t; q)}{\partial x} = \frac{\partial^2 \phi(x, t; q)}{\partial x^2} - \phi^2(x, t; q)$$

We gain

$$u_0(x, t) = e^{-x},$$

$$u_1(x, t) = \frac{t}{1!}e^{-x},$$

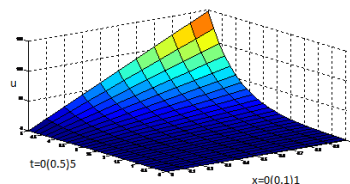


Figure 1: The surface area shows $u(x, t)$ for Ex.(1) for $x = 0(0.1)1$ and $t = 0(0.5)5$

$$u_2(x, t) = \frac{t^2}{2!} e^{-x},$$

$$u_3(x, t) = \frac{t^3}{3!} e^{-x} \dots$$

The final solution is

$$u(x, t) = e^{t-x}$$

The surface area shows $u(x, t)$ for $x = (0, 100)$ and $t = (0, 100)$

Example. 3

Consider the time fractional advection dispersion equation

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = D \frac{\partial^2 u(x, t)}{\partial x^2} - b \frac{\partial u(x, t)}{\partial x} + \lambda u(x, t), \quad t > 0, 0 < \alpha \leq 1 \quad (11)$$

Where $D = 2, b = 2$ and $\lambda = 1$
with initial condition

$$u(x, 0) = e^x \quad (12)$$

Now we will apply the HAM to solve Eq.(11) subject to the initial condition Eq.(12)

We define the nonlinear operator as

$$N[\phi(x, t; q)] = \frac{\partial^\alpha \phi(x, t; q)}{\partial t^\alpha} - 2 \frac{\partial^2 \phi(x, t; q)}{\partial x^2} + 2 \frac{\partial \phi(x, t; q)}{\partial x} - \phi(x, t; q) \quad (13)$$

and linear operator $L[\phi(x, t; q)] = \frac{\partial^\alpha \phi(x, t; q)}{\partial t^\alpha}$

With the property $L[C_1(x)] = 0$ Where $C_1(x)$ is the integration constant.

Now by using Eq.(7) we have

$$\mathfrak{R}_m(\vec{U}_{m-1}) = N[\phi(x, t; q)] = \frac{\partial^\alpha \phi(x, t; q)}{\partial t^\alpha} - 2 \frac{\partial^2 \phi(x, t; q)}{\partial x^2} + 2 \frac{\partial \phi(x, t; q)}{\partial x} - \phi(x, t; q)$$

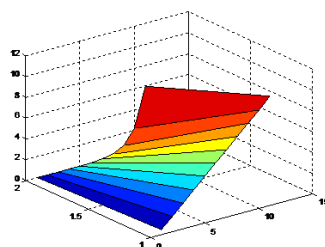
and the solution of the m^{th} order deformation Eq.(11) for $m \geq 1$ becomes

$$u_{m-1}(x, t) = \chi_m U_{m-1}(x, t) + h H(r, t) L^{-1}[\mathfrak{R}_m(\vec{U}_{m-1})]$$

since $m \geq 1$, $\chi_m = 1$, we set $h = -1$ and $H(r, t) = 1$ we obtain the final solution as

$$u(x, t) = \sum_{k=0}^{\infty} e^x \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}$$

The surface shows the solution $u(x, t)$ for $\alpha = 1$ and $t = 0.5$

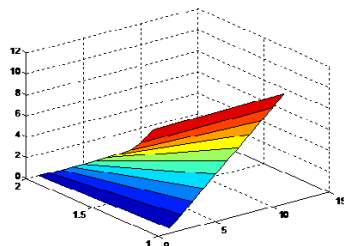


The surface shows the solution $u(x, t)$ for $\alpha = 1$ and $t = 1.5$

Example. 4

Consider another time fractional advection dispersion equation

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = D \frac{\partial^2 u(x, t)}{\partial x^2} - b \frac{\partial u(x, t)}{\partial x} + \lambda u(x, t), \quad t > 0, 0 < \alpha \leq 1 \quad (14)$$



Where $D = 1, b = 1$ and $\lambda = 1$
with initial condition

$$u(x, 0) = \sin x \quad (15)$$

Now we will apply the HAM to solve Eq.(11) subject to the initial condition Eq.(15)

We define the nonlinear operator as

$$N[\phi(x, t; q)] = \frac{\partial^\alpha \phi(x, t; q)}{\partial t^\alpha} - \frac{\partial^2 \phi(x, t; q)}{\partial x^2} + \frac{\partial \phi(x, t; q)}{\partial x} - \phi(x, t; q) \quad (16)$$

and linear operator $L[\phi(x, t; q)] = \frac{\partial^\alpha \phi(x, t; q)}{\partial t^\alpha}$

With the property $L[C_1(x)] = 0$ Where $C_1(x)$ is the integration constant.

Now by using Eq.(7) we have

$$\mathfrak{R}_m(\vec{U}_{m-1}) = N[\phi(x, t; q)] = \frac{\partial^\alpha \phi(x, t; q)}{\partial t^\alpha} - \frac{\partial^2 \phi(x, t; q)}{\partial x^2} + \frac{\partial \phi(x, t; q)}{\partial x} - \phi(x, t; q)$$

and the solution of the m^{th} order deformation Eq.(11) for $m \geq 1$ becomes

$$u_{m-1}(x, t) = \chi_m U_{m-1}(x, t) + h H(r, t) L^{-1}[\mathfrak{R}_m(\vec{U}_{m-1})]$$

since $m \geq 1$, $\chi_m = 1$, we set $h = -1$ and $H(r, t) = 1$

and also the equation has super diffusive behaviour we obtain the

final solution as

$$u(x, t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k\alpha}}{\Gamma(2k\alpha + 1)} - \cos x \sum_{k=0}^{\infty} (-1)^k \frac{t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha + 1)}$$

Figure 2: Plot of $u(x, t)$ vs. h for $\alpha = 1$ and $t = 0.5$

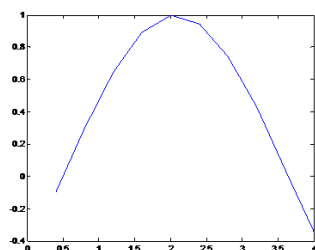
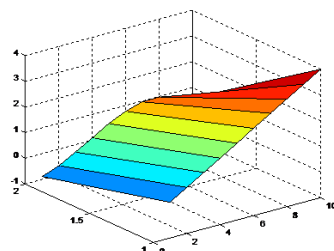


Figure 3: The surface shows the solution $u(x, t)$ for $\alpha = 1$ and $t = 0.5$



3 Conclusion

The homotopy analysis method (HAM) is applied to obtain the approximate/analytical solutions for a class of PDEs arising in mathematical chemistry. It avoids the need for calculating the Adomian

polynomials which can be difficult in some cases. The reliability of the method and reduction in the size of computational domain give this method wider applicability. The results show that HAM is a powerful mathematical tool for finding the exact and approximate solutions of the nonlinear equations. In addition the calculations involved in HAM are simple, straight forward and low computation cost.

Acknowledgement

This work was supported by the Department of Science and Technology (DST-SERB, Project No.: SB/FTP/MS-012/2013), Government of India. Our hearty thanks are due to Prof. R. Sethuraman, Vice-Chancellor, SASTRA University, Dr. S. Vaidhyasubramaniam, Dean / Planning and development and Dr. S. Swaminathan, Dean / Sponsored research for their kind encouragement and for providing good research environment.

References

- [1] Abbasbandy. S, The application of homotopy analysis method to nonlinear equations arising in heat transfer, *Phy.Letters.A.*, **360** (2006), 109-113.
- [2] Abbasbandy. S, Soliton solutions for the FitzhughNagumo equation with the homotopy analysis method, *Appl. Math. Modelling.*, **32** (2008), 2706-2714.
- [3] Bizzar. J, Ghazvini. H, Convergence of the homotopy perturbation method for partial differential equations, *Nonlinear analysis, Real world applications.*, **10** (2009), 2633-2640.
- [4] Domairy. G, Nadi. N, Assessment of homotopy analysis method and homotopy perturbation method in non linear heat transfer equation, *Int. J. Commun. heat and mass transfer.*, **35** (2008), 93-102.
- [5] Golbabai. A, Javidi. M.A, Variational Iteration method for solving parabolic partial differential equations, *Comp. Math. Appl.*, **54** (2007), 987-982.

- [6] Hariharan. G, The homotopy analysis method applied to the Kolmogorov-Petrovskii-Piskunov (KPP) and fractional KPP equation, *J Math. Chem.*, **51** (2013), 992-1000.
- [7] Hariharan. G, Rajaraman. R, A new coupled wavelet-based method applied to the nonlinear reaction-diffusion equation arising in mathematical chemistry, *J Math. Chem.*, **51** (2013), 2386-2400.
- [8] Hariharan. G, Kannan. K, Haar wavelet method for solving some nonlinear parabolic equations, *J Math. Chem.*, **48** (2010), 1044-1061.
- [9] Hashim. I, Noorani. M.S.M, Said Al-Hadidi. M.R, Solving the generalized Burgers-Huxley equation using the Adomian decomposition method, *Math. Comp. Model.*, **43** (2001), 1404-1411.
- [10] Jafari. H, Zabihi. M, Said. M, Application of homotopy perturbation method for solving gas dynamics equation, *Appl. Math. Sci.*, **2** (2008), 2393-2396.
- [11] Kucukarslan. S, Homotopy perturbation method for coupled Schrödinger-KdV equation, *Nonlinear Analysis: Real World Applications.*, **10** (2009), 2264-2271.
- [12] Kazemi-Sangsereki. M, Ganji. D.D, Gorg. M, Application of homotopy perturbation method and variational iteration method to gas dynamics equation, *J App. Eng. Math.*, **1** (2011), 27-34.
- [13] Liao. S.J, Comparison between the Homotopy analysis method and Homotopy perturbation method, *Appl. Math. Comp.*, **169** (2005) 118-164.
- [14] Liao. S.J, Notes on Homotopy analysis method: Some definitions and theorems *Comm. Nonlinear Sci.Numer.Simul.*, **14** (2009) 983-997.
- [15] Liao. S.J, On the Homotopy analysis method for non linear problems, *Appl. Math. Comp.*, **147** (2004) 499-513.

- [16] Liao. S.J, Beyond perturbation: Introduction to Homotopy Analysis Method, *Chapman Hall/CRC press. Boca Raton* (2003).
- [17] Matinfar. M, Raeisi. Z, Variational Homotopy perturbation method for solving the nonlinear gas dynamics equation, *Int. J. Math. Modelling Comput.*, **1**(3) (2011) 183-187.
- [18] Molabahrami. A, Khani. F, The homotopy analysis method to solve Burgers-Huxley equation, *Non linear analysis: Real world Applications.*, **10** (2009) 589-600.
- [19] Yldrm. A, Application of Hes homotopy perturbation method for solving the Cauchy reactiondiffusion problem, *Comp. Math. App.*, **57**(4) (2009) 612-618.

