

Vertex-edge and Connected domination numbers of a tree

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November 8, 2017

Abstract

A set $D \subseteq V(G)$ of a graph G is a dominating set if every vertex of $V(G) - D$ has a neighbor in D . A dominating set D is connected dominating set(CDS) if the induced subgraph $\langle D \rangle$ is connected. The connected domination number of a graph G , denoted by $\gamma_c(G)$, is the minimum cardinality of a CDS of G . A vertex-edge dominating set(VEDS) of a graph G is a set D of vertices of G such that every edge of G is incident with a vertex of D or a vertex adjacent to a vertex of D . The vertex edge domination number of a graph G , denoted by $\gamma_{ve}(G)$, is the minimum cardinality of a VEDS of G . We characterize the family of trees for which $\gamma_{ve}(T) = \gamma_c(T)/2$.

AMS Subject Classification: 05C69.

Key Words and Phrases:vertex-edge domination, connected domination, tree.

1 Introduction

A graph G with no loops and multiple edges are considered. Two vertices u and v are adjacent if there is an edge incident with u and v . The number of vertices adjacent to a vertex w is called degree of the vertex w and is denoted by $d_G(w)$. A vertex adjacent to exactly one vertex is called pendent vertex and the edge incident with the pendent vertex is called end edge. The vertex adjacent to pendent vertex is called a support vertex. The support vertex is weak(strong, respectively) if it is adjacent to exactly one leaf(more than one pendent vertex, respectively). Let P_n be a path having n vertices. Two vertices are connected if there is a path between the vertices. A set $D \subseteq V(G)$ is connected if there exists a path between any two vertices of D . The complete bipartite graph $K_{1,n}$ is called a star and its centre is the vertex of degree > 1 in $K_{1,n}$. The double star is a tree obtained by joining the centres of two stars by an edge.

A subset $D \subseteq V(G)$ is a dominating set(DS) of G if every vertex $v \in V(G) \setminus D$ has a neighbour in D . The domination number of a graph G , denoted by $\gamma(G)$, is the minimum cardinality of a DS of G . The reader is referred to [3] for a detailed survey on domination in graphs. A dominating set D of a graph G is connected dominating set(CDS) if the induced subgraph $\langle D \rangle$ is connected. The minimum cardinality of a CDS of G is the connected domination number of G and denoted by $\gamma_c(G)$. A CDS of G of minimum cardinality is called a $\gamma_c(G)$ -set. E. Sampathkumar introduced the idea of Connected domination [7] in graphs, and was further studied in [1, 2].

An edge $e \in E(G)$ is said to be vertex-edge dominated (ve-dominated, in short) by a vertex $v \in V(G)$ if e is incident at v , or e is adjacent to an edge incident at v . A subset $D \subseteq V(G)$ is a vertex-edge dominating set(VEDS) of G if every edge of G is ve-dominated by a vertex of D . The vertex-edge domination number of G , denoted by $\gamma_{ve}(G)$, is the minimum cardinality of a VEDS of G . A VEDS of G with minimum cardinality is called a $\gamma_{ve}(G)$ -set. In [6], vertex-edge domination, a variety of mixed domination was introduced, and was further studied in [4, 5].

Equality of connected domination with other domination numbers is an interesting study. To mention a few, equality of connected domination number and domination numbers was

characterized in [1] for trees. Characterization of trees with equal connected domination and total domination numbers are given in [8]. Two of the authors have characterized connected domination number equal to chromatic transversal number for trees in [2]. In this paper, we prove $\gamma_{ve}(T) \leq \gamma_c(T)/2$ for trees with $diam(T)$ above two and trees attaining this bound is characterized.

2 Results

A tree with diameter atmost two, $\gamma_c(T) = \gamma_{ve}(T) = 1$. Therefore throughout our discussion we consider trees T with diameter atleast three.

First we show that for any nontrivial tree T of order n , $\gamma_c(T)/2$ is an upper bound of $\gamma_{ve}(T)$. In order to characterize the trees attaining this bound we construct a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 = P_4$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k .
- Operation $\mathcal{O}_2$: Attach a support vertex of P_4 to a vertex of T_k adjacent to a support vertex of P_4 or to a path P_3 or a path P_2 .
- Operation $\mathcal{O}_3$: Attach a path P_3 by joining one of its leaves to a vertex of T_k adjacent to a path P_3 or P_2 .

We shall show that for every tree T of the family $\mathcal{T}$ has $\gamma_{ve}(T) = \gamma_c(T)/2$.

Lemma 1. *If T belongs to $\mathcal{T}$, then $\gamma_{ve}(T) = \gamma_c(T)/2$.*

Proof. We prove this result using the principle of induction on the number of operations k performed to construct the tree T . If $T = P_4$, then $\gamma_c(T) = 2$ and $\gamma_{ve}(T) = 1$. We have $\gamma_c(T)/2 = 1 = \gamma_{ve}(T)$. Let k be a positive integer. Assume that the lemma is true for all tree $T' = T_k$ belonging to the family $\mathcal{T}$ constructed by $k - 1$ operations. The number of vertices of the tree T' is denoted

as n' . Let T be a member of the family $\mathcal{T}$ obtained from T_1 by k operations.

By operation $\mathcal{O}_1$, let T be constructed from T' . One can easily see that any $\gamma_{ve}(T')$ -set is an VEDS of T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. Also, it is obvious that $\gamma_{ve}(T') \leq \gamma_{ve}(T)$. Therefore $\gamma_{ve}(T) = \gamma_{ve}(T')$. It is easy to observe that $\gamma_c(T) = \gamma_c(T')$. Thus, we get $\gamma_{ve}(T) = \gamma_{ve}(T') = \gamma_c(T')/2 = \gamma_c(T)/2$.

By operation $\mathcal{O}_2$, let T be constructed from T' . Let x be a vertex of T' to which a support vertex v_2 of a path $P'_4 : v_1v_2v_3v_4$ is attached. Let $P_4 : u_1u_2u_3u_4$ be a path different from P'_4 . Let the vertex u_2 of P_4 be adjacent to x . Let A be a $\gamma_{ve}(T)$ -set. The edges v_3v_4, u_3u_4 are dominated by the vertices $v_2, u_2 \in A$. It is clear that $A \setminus \{v_2\}$ is a VEDS of T . Therefore $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. Let A' be a $\gamma_{ve}(T')$ -set. It is easy to observe that $A' \cup \{v_2\}$ is a VEDS of the tree T and so $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Hence $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$. Let S denote a $\gamma_c(T)$ -set of the tree T . The vertices v_4, v_1, u_4 and u_1 are dominated by the vertices $v_3, v_2, u_3, u_2 \in S$. Since $\langle S \rangle$ is connected, the vertex $x \in S$. It is easy to observe that $S \setminus \{v_2, v_3\}$ is a CDS of T' . Thus $\gamma_c(T') \leq \gamma_c(T) - 2$. Let S' denote a $\gamma_c(T')$ -set. Then clearly $S' \cup \{v_2, v_3\}$ is a CDS of the tree T . Thus $\gamma_c(T) \leq \gamma_c(T') + 2$. This proves that $\gamma_c(T') = \gamma_c(T) - 2$. We now get $\gamma_{ve}(T) = \gamma_{ve}(T') + 1 = \gamma_c(T')/2 + 1 = \gamma_c(T)/2$.

Let the vertex x be adjacent to the vertex a of the path $P_3 = abc$ and the vertex v_2 of the path $P_4 : v_1v_2v_3v_4$. Let A' be a $\gamma_{ve}(T')$ -set. It is easy to verify that $A' \cup \{v_2\}$ is a VEDS of T' . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let A be a $\gamma_{ve}(T)$ -set. The edges v_3v_4, bc , are dominated by the vertices $v_2, a \in D$. Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. This implies that $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$. Let S be a $\gamma_c(T)$ -set. The vertices v_4, v_1 and c are dominated by the vertices $v_3, v_2, b \in S$. The subgraph induced by S denoted by $\langle S \rangle$ is connected, the vertices $x, a \in S$. So, it is clear that $D \setminus \{v_2, v_3\}$ is a CDS of the tree T' . Thus $\gamma_c(T') \leq \gamma_c(T) - 2$. Let S' be a $\gamma_c(T')$ -set of T' . It is evident that $S' \cup \{v_2, v_3\}$ is a CDS of T . Thus $\gamma_c(T) \leq \gamma_c(T') + 2$. This evidently implies that $\gamma_c(T) = \gamma_c(T') + 2$. We now get $\gamma_{ve}(T) = \gamma_{ve}(T') + 1 = \gamma_c(T')/2 + 1 = \gamma_c(T)/2$.

Now assume that the vertex x is adjacent to a path $P_2 = ab$. Arguing as above, we get $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$ and $\gamma_c(T') = \gamma_c(T) - 2$. Thus $\gamma_{ve}(T) = \gamma_{ve}(T') + 1 = \gamma_c(T')/2 + 1 = \gamma_c(T)/2$.

By operation $\mathcal{O}_3$, let T be constructed from T' . Let x be a vertex to which a path P_3 is attached. Let $u_1u_2u_3$ be the attached path. Let u_1 be joined to x . Let $v_1v_2v_3$ be another path different from $u_1u_2u_3$ adjacent to x . Let v_1 be joined to x . Let A be a $\gamma_{ve}(T)$ of the tree T . The edges v_2v_3, u_2u_3 are dominated by the vertices $v_1, u_1 \in A$. It is obvious that $A \setminus \{u_1\}$ is a VEDS of the tree T' . Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. Let A' be a $\gamma_{ve}(T')$ -set. It is easy to see that $A' \cup \{u_1\}$ is a VEDS of T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. This implies that $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$. Let S be a $\gamma_c(T)$ -set. To dominate the vertices v_3, u_3 the vertices $v_2, u_2 \in S$. The set S is connected, the vertices $u_1, x, v_1 \in S$. It is easy to see that $S \setminus \{u_1, u_2\}$ is a CDS of T' . Thus $\gamma_c(T') \leq \gamma_c(T) - 2$. Let S' be a $\gamma_c(T')$ -set. It is obvious that $S' \cup \{u_1, u_2\}$ is a CDS of T . Thus $\gamma_c(T) \leq \gamma_c(T') + 2$. This clearly shows that $\gamma_c(T) = \gamma_c(T') + 2$. We have $\gamma_{ve}(T) = \gamma_{ve}(T') + 1 = \gamma_c(T')/2 + 1 = \gamma_c(T)/2$. $\square$

We now give an upper bound on the vertex-edge domination number of a tree in terms of its connected domination number and the extremal trees attaining this bound is characterized.

Theorem 2. *If T is a tree of order n with diameter at least three, then $\gamma_{ve}(T) \leq \gamma_c(T)/2$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $\text{diam}(T) = 1$, then $T = P_2$ and so $\gamma_c(T)/2 = \gamma_{ve}(T)$. If $\text{diam}(T) = 2$, then T is a star. This gives $\gamma_c(T)/2 = \gamma_{ve}(T)$. Let $\text{diam}(T) = 3$. If $T = P_4$, then $\gamma_t(T) = 2$ and $\gamma_{ev}(T) = 1$. Thus $T \in \mathcal{T}$. Let T be a double star other than P_4 , we get T from P_4 by applying the first operation on support vertices, a finite number times. Thus $T \in \mathcal{T}$.

Now assume that $\text{diam}(T) \geq 4$. Thus the order n of the tree T is at least five. We obtain the result by the principle of induction on n . Assume that the theorem is true for every tree T' of order $n' < n$.

First assume that some support vertex of T , say x , is strong. Let y be a leaf adjacent to x . Let $T' = T - y$. Let D' be a $\gamma_{ve}(T')$ -set. It is obvious that D' is a VEDS T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. Now let S be a $\gamma_c(T)$ -set. It is obvious that S is a CDS of the tree T' . Thus $\gamma_c(T') \leq \gamma_c(T)$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') = \gamma_c(T')/2 \leq \gamma_c(T)/2$. If $\gamma_{ve}(T) = \gamma_c(T)/2$, then $\gamma_{ve}(T') = \gamma_c(T')/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T can be obtained from T'

by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r , v be the parent of t , u be the parent of v and w be the parent of u in the rooted tree. If $\text{diam}(T) \geq 5$, then let d be the parent of w . If $\text{diam}(T) \geq 6$, then let e be the parent of d . We denote by T_x the sub-tree induced by a vertex x and its descendants in the rooted tree T .

Assume that there is a support vertex x , other than v , adjacent to u . Let y be the child of x . Let $T' = T - T_v$. Let A' be a $\gamma_{ve}(T')$ -set. It is obvious that $u \in A'$. The vertex u dominates the edges uv, vt . Therefore, D' is a VEDS of T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. Let S be a $\gamma_c(T)$ -set. It is clear that $u, v \in S$. It is seemingly true that $S \setminus \{v\}$ is a CDS of T' . Thus $\gamma_c(T') \leq \gamma_c(T) - 1$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') = \gamma_c(T')/2 \leq (\gamma_c(T) - 1)/2 < \gamma_c(T)/2$.

Let x be a leaf and child of u . Now assume that the vertex a is the child of w different from u with the farthest distance of w to a vertex in T_a is two or three. It equivalent to consider the case T_k is isomorphic to T_u or P_3 or P_2 . Let $T' = T - T_u$. Let DA' be a $\gamma_{ve}(T')$ -set. It is easy to see that $A' \cup \{u\}$ is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let S be a $\gamma_c(T)$ -set. The vertices t, x are dominated by the vertices $v, u \in S$. It is true that $S \setminus \{u, v\}$ is a CDS of T' . Thus $\gamma_c(T') \leq \gamma_c(T) - 2$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 = \gamma_c(T')/2 + 1 \leq \gamma_c(T)/2$. If $\gamma_{ve}(T) = \gamma_c(T)/2$ then $\gamma_{ve}(T') = \gamma_c(T')/2$. By the induction hypothesis $T' \in \mathcal{T}$. We get the tree T from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Assume $d_T(w) = 3$. Let a be the leaf adjacent to w . Let $T' = T - T_w$. Let A' be a $\gamma_{ve}(T')$ -set. One can easily see that $A' \cup \{u\}$ is a VEDS of T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let S be a $\gamma_c(T)$ -set. The vertices t, x and a are dominated by the vertices $v, u, w \in S$. Clearly, $S \setminus \{u, v, w\}$ is a CDS of T' . Thus $\gamma_c(T') \leq \gamma_c(T) - 3$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 = \gamma_c(T')/2 + 1 \leq (\gamma_c(T) - 1)/2 < \gamma_c(T)/2$.

Assume $d_T(w) = 2$. Let $d_T(d) = 1$. We have $T = P_5$ with a vertex adjacent to center of P_5 . We get $\gamma_{ve}(T) = 1 < 3 = \gamma_c(T)$. Let $d_T(d) \geq 2$. Let $T' = T - T_w$. Let A' be a $\gamma_{ve}(T')$ -set. It is clear that $A' \cup \{u\}$ is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let S be a $\gamma_c(T)$ -set. The vertices t, x are dominated by the vertices $v, u \in S$. Since $d_T(d) \geq 2$ and $\langle S \rangle$ is connected, the vertex $w \in S$. It is seemingly true that $S \setminus \{v, u, w\}$ is a CDS of the tree T' . Thus

$\gamma_c(T') \leq \gamma_c(T) - 3$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 = \gamma_c(T')/2 + 1 \leq (\gamma_c(T) - 1)/2 < \gamma_c(T)/2$.

Now assume $d_T(u) = 2$. Assume that w is adjacent to P_3 or P_2 . Let $T' = T - T_u$. Let A' be a $\gamma_{ve}(T')$ -set. It is clear that $A' \cup \{u\}$ is a VEDS of T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let S be a $\gamma_c(T)$ -set. It is obvious that $u, v, w \in S$. It is clear that $S \setminus \{v, u\}$ is a CDS of T' . Thus $\gamma_c(T') \leq \gamma_c(T) - 2$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 = \gamma_c(T')/2 + 1 \leq \gamma_c(T)/2$. If $\gamma_{ve}(T) = \gamma_c(T)/2$ then $\gamma_{ve}(T') = \gamma_c(T')/2$. By the induction hypothesis $T' \in \mathcal{T}$. We get the tree T from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume $d_T(w) = 3$. Let x be a leaf adjacent to w . Let $d_T(d) = 1$. We have $T = P_5$ with a support vertex of degree three. This gives $\gamma_{ve}(T) = 1 < 3/2 = \gamma_c(T)/2$. Now assume that $d_T(d) \geq 2$. Let $T' = T - T_w$. Let A' be a $\gamma_{ve}(T')$ -set. It is clear that $A' \cup \{u\}$ is a VEDS of T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let S be a $\gamma_c(T)$ -set. The vertices t, x are dominated by the vertices $v, w \in S$. Since $\langle S \rangle$ is connected, the vertex $u \in S$. It is true that $S \setminus \{v, u, w\}$ is a CDS of T' . Thus $\gamma_c(T') \leq \gamma_c(T) - 3$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 = \gamma_c(T')/2 + 1 \leq (\gamma_c(T) - 1)/2 < \gamma_c(T)/2$.

Assume that $d_T(d) \geq 2$. Let $T' = T - T_w$. Let A' be a $\gamma_{ve}(T')$ -set. It is clear that $A' \cup \{u\}$ is a VEDS of T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let S be a $\gamma_c(T)$ -set. We require the vertices $v, w \in S$ so as to dominate the vertices t, x . Since $\langle S \rangle$ is connected, the vertex $u \in S$. It is true that $D \setminus \{v, u, w\}$ is a CDS of T' . Thus $\gamma_c(T') \leq \gamma_c(T) - 3$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 = \gamma_c(T')/2 + 1 \leq (\gamma_c(T) - 1)/2 < \gamma_c(T)/2$. $\square$

Acknowledgement: The first author's research is supported by National Board of Higher Mathematics, India and also the authors thank DST FIST for the support to the Department of Mathematics, SASTRA University through Grant No. SR/FST/MSI-107/2015.

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