

On separation axioms and continuity with respect to some types of sets in ideal topological space

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Abstract:-

The purpose of this work is to define the feebly – ideal – separation axiom, and study the relation between these axioms and some types of separation axioms in other spaces. , also we define and study feebly – ideal– continuous function with some theorems and relations between this type and others which are defined in other spaces.

Keywords: *Feebly continuous, Ideal – continuous, feebly – ideal – continuous, feebly – ideal open function, feebly – ideal – closed function, feebly – ideal – separation axioms.*

I. Introduction:-

The concept of ideal topology in the classic text was introduced by kuratowski [7], D. Jankovie and, R Hamlelt [5] . Introduced the concept of I – open set in ideal Topological space after that M. E. Abd El. Monsef , E. F. Iashien and A. A. Nasef [8] introduced a new study about the I – open set,, in 2017 [12] Yiezi K. and suadd H. introduced the concept of feebly – ideal – open set which is defined for every subset A of X , $A \subseteq Scl(int(A^*))$ where A^* is a local function of A . In this paper we define the Feebly – ideal – separation axioms with some theorems and properties and then some relations between them and another types of separation axioms like $I-T_i$ –(respectively feebly- T_i) separation axiom where $i=0,1,2,3,4$ which are defined by [1, 9 , 13]by replacing the open sets in topological space by I -open set(respectively , feebly-open set)all the relations are discussed, also we defined the feebly – ideal – continuous and feebly – ideal open (rep. closed) function with some of theorem and relation on it . in this paper X_{F-I} it mean the set of all the Feebly – ideal – open set in X , F – continuous it is mean Feebly – continuous function also an ideal topological space (X,T,I) is called T -*-dense space (respectively , T - T^* -closed space) if every T -open set is *-dense set (respectively T^* -closed set)

II - Preliminaries

Remark (2-1) [9] :-

In the ideal topological space (X, T, I) if X is $I - T_{i+1}$ then X is $I - T_i$ when $i = 0, 1, 2, 3$.

Theorem (2-2) [1] :-

In the topological space (X, T) if X is Feebly - T_{i+1} , then X is Feebly - T_i when $i = 0, 1, 2, 3$.

Remark (2-3)[9] :-

In the ideal topological space (X, T, I) , $I - T_i$ - space and T_i - space independent. Where $i = 0, 1, 2, 3, 4$.

All the above relations are clarified in the following diagram (1) as follow.

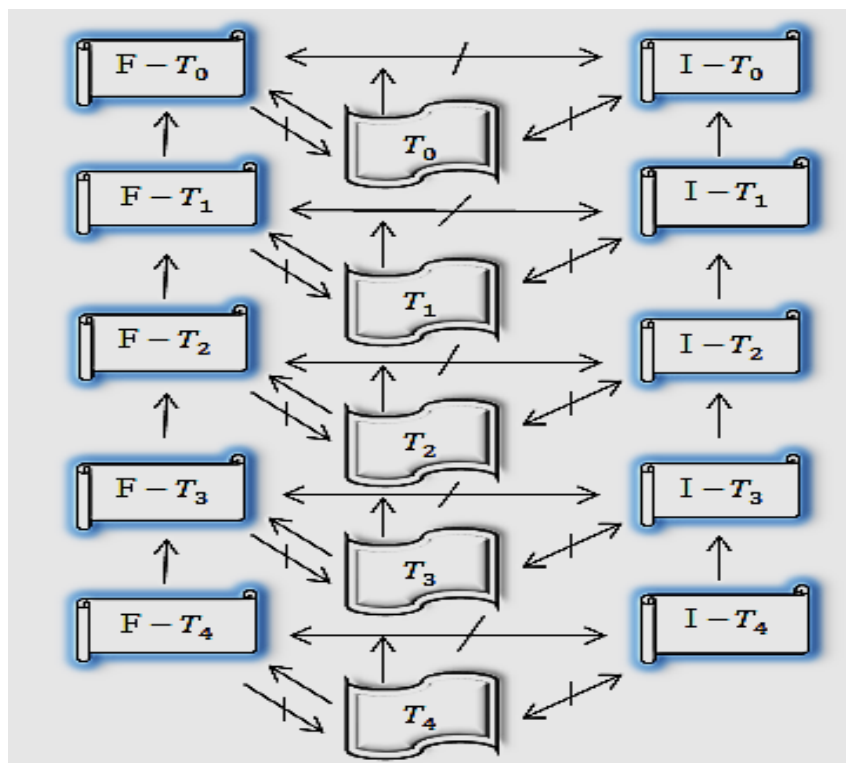


Diagram (1)

Theorem (2-4):-

In the ideal topological space (X, T, I) Every T_i space is $I - T_i$ space if X is T^* -dense space.

Theorem (2-5):-

In the ideal topological space (X, T, I) Every $I - T_i$ space is T_i space if X is $T - T^*$ - closed space

.Definition (2-6) [8] :-

A function $f: (X, T, I) \rightarrow (Y, \sigma)$ is said to be I - continuous if for every σ - open set V , $f^{-1}(V)$ is I - O set in X .

Remark (2-7) [8] :-

The concept of continuity and I - continuity are independent.

Definition (2-8) [2] :-

A function $f: (X, T) \rightarrow (Y, V)$ is said to be feebly - continuous if the inverse image by f of each open set H of Y is feebly open in X .

Theorem (2-9) [2] :-

Every continuous mapping is feebly continuous mapping.

Remark (2-10):-

The concept of ideal - continuity and feebly - continuity are independent.

Example (2-11) :-

Let $X = \{1, 2, 3, 4\}$ and $T = \{\emptyset, X, \{1\}, \{1, 2, 3\}\}$ and let $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $f: X \rightarrow Y$ defined by $f(1) = f(2) = a$, $f(3) = f(4) = b$, the mapping f is feebly - continuous but not ideal - continuous.

Definition (2-12) [2] :-

A function $f: X \rightarrow Y$ is called feebly - open (resp. feebly - closed) if the image of each open (resp. closed) set in X is feebly - open (resp. feebly - closed) in Y .

Theorem (2-13) [2] :-

1) Every open mapping is feebly open mapping and the converse is not necessarily true.

2) Every closed mapping is feebly – closed mapping and the converse is not necessarily true.

Definition (2-14) [8]:-

A function $f: (X, T) \rightarrow (Y, d, J)$ is called I – open (pres. I – closed) if the image of each open (resp. closed) set in X is ideal – open (resp. closed ideal) set in Y .

Remark (2-15):-

1) Each of I – open function and open function are independent. [8]

2) The concept of I – open function and feebly – open function are independent.

III - On Feebly – ideal – separation axioms

Definition (3-1) :-

Let (X, T, I) be a space with topological T and I be an – ideal on X then X is called:-

1) Feebly – ideal – T_0 space :- if for each two distinct point x, y of X there exist Feebly – ideal – open set U of X such that U containing one of this point but not containing the other, and we denoted by $F - I - T_0$ space .

2) Feebly – ideal – T_1 space :- if for each two distinct point x, y there are Feebly – ideal – open set U, V such that $x \in U$ and $x \notin V$ and $y \in V$ and $y \notin U$ and we denoted by $F - I - T_1$ space .

3) Feebly – ideal – T_2 space :- if for each two distinct point x, y there exists disjoint Feebly – ideal – open set U, V of X such that $x \in U$ and $y \in V$, and we denoted by $F - I - T_2$ space .

4) Feebly – ideal – regular space :- if for each point x in X and H is closed subset of X such that $x \notin H$ there exists disjoint Feebly – ideal – open set U, V of X such that $x \in U, H \subset V$ and if X is Feebly – ideal – regular space and Feebly – ideal . T_1 Space then X is feebly – ideal – T_3 space.

5) Feebly – ideal – normal space :- if for each A, B of disjoint closed subset of X , there are a pair U, V of disjoint Feebly – ideal – open sets of X such that $A \subseteq U$ and $B \subseteq V$ and if X is Feebly – ideal – normal space and Feebly – ideal - T_1 space . Then X is feebly – ideal – T_4 space we denoted by $F - I - T_4$ space.

Theorem (3-2) :-

Let (X, T, I) is ideal Topological then X is Feebly – ideal – T_1 space if and only if $f - \{x\}$ is closed, for each x on X .

Proof:-

Let (X, T, I) is Feebly – ideal – T_1 space let $x \in X$ if $y \in X$, such that x, y are disjoint point, since X is Feebly – ideal – T_1 space, then there exist U is Feebly – ideal – open set, such that $y \in U$ and $x \notin U$, $y \notin f - I - cl\{x\}$, and then $f - I - cl\{x\} \subseteq \{x\}$, and hence $\{x\} = f - I - cl\{x\}$.

Proposition (3-3) :-

In the ideal Topological Space (X, T, I) if X is $F - I - T_1$. Then X is $F - I - T_0$.

Proof:-

Let (X, T, I) is a $F - I - T_1$, and let $x, y \in X$ such that $x \neq y$, Since X is a $F - I - T_1$, then there exist an $U, V \in X_{F-I}$, Such that $x \in U$ but $y \notin U$ and $y \in V$, $x \notin V$, and then x is a $F - I - T_0$.

Proposition (3-4) :-

In the ideal Top. Space (X, T, I) if X is $F - I - T_2$. Then x is $F - I - T_1$.

Proof:-

Let (X, T, I) is a $F - I - T_2$, and let $x, y \in X$ such that $x \neq y$, Since X is a $F - I - T_2$, then there exist an $U, V \in X_{F-I}$, Such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$, Since $U \cap V = \emptyset$, then $x \in U$, $x \notin V$ or $y \in V$, $y \notin U$, and then x is a $F - I - T_1$.

Proposition (3-5):-

In the ideal Topological Space (X, T, I) if X is $F - I - T_3$ space and (X, T) is T_1 space then X is $F - I - T_2$ space.

Proof:-

Let x, y are two distinct points of the ideal Topological space X , Since X is T_1 Space, then $\{x\}$ is closed $y \notin \{x\}$ since X is $F - I - T_3$ space, Then X is $F - I -$ regular, then there exist two distant open set $F - I -$ open set U, V , such that $y \in U$, $\{x\} \subseteq V$, then $y \in U$, $x \in V$ and $U \cap V = \emptyset$, then X is $F - I - T_2$ space.

Proposition (3-6):-

In the ideal Topological Space (X, T, I) if X is $F-I-T_4$ space and (X, T) is T_1 space then X is $F-I-T_3$ space .

Proof :-

Let X is $F-I-T_4$ space , and let $x \in X$, H is closed set such that $x \notin H$, Since X is a $F-I-T_4$, then X is $F-I-T_1$ space and $F-I$ - normal, Since X is T_1 space then $\{x\}$ is closed, hence $\{x\} \cap H = \emptyset$, since X is $F-I$ - normal, Then there exist $U, V \in \mathcal{F}_{F-I}$ such that $\{x\} \subseteq U, H \subseteq V, U \cap V = \emptyset$, Then $x \in U, H \subseteq V, U \cap V = \emptyset$, then X is $F-I$ - reg. and X is $F-I-T_1$, Then X is $F-I-T_3$.

The following diagrams implication gives the relationship among Feebly – ideal – T_i space when $i = 0,1,2,3,4$.

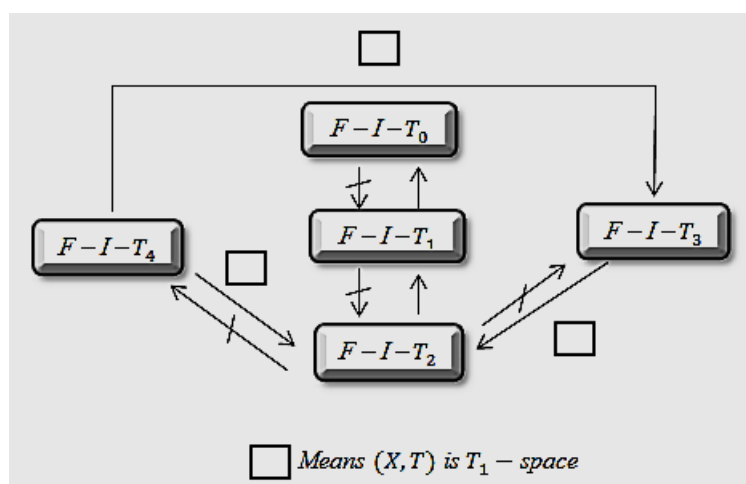


Diagram (2)

Remark (3-7) :-

Let (X, T, I) is ideal Topological space when $i = 0, 1, 2, 3, 4$ then :

- 1) $F-I-T_i$ space and T_i space are independent.
- 2) $F-I-T_i$ space and $I-T_i$ space are independent.
- 3) $F-I-T_i$ space and $F-T_i$ space are independent.

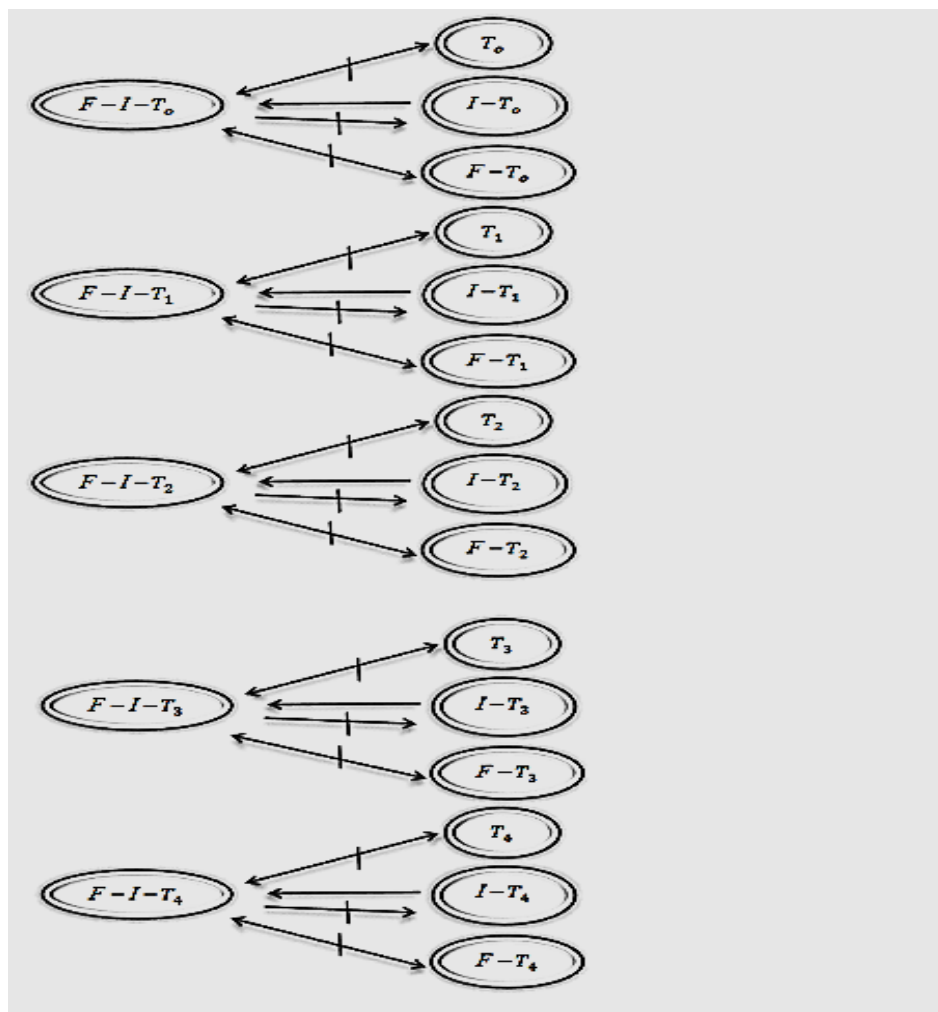
Remark (3-8) :-

In the ideal Topological space (X, I, T) if X is $I - T_i$ - space then X is $F - I - T_i$ space when $i = 0, 1, 2, 3, 4$.

Proof:-

Exist by Definition and by proposition (3.5).

The following diagrams implication gives us the relationship among Feebly - ideal - T_i space, Feebly - T_i space, ideal - T_i space and T_i space.



Diagrams (3)

IV – On feebly – ideal – continuous functionsDefinition (4-1) :-

A function $f: (X, T, I) \rightarrow (Y, \sigma)$ is said to be feebly – ideal – continuous if the inverse image of each open set H of Y is feebly – ideal – open in X .

Remark (4-2) :-

feebly – continuous and feebly – ideal – continuous are independent as we show in the following example

Example (4-3) :-

Let $f: (X, T, I) \rightarrow (Y, \sigma)$, $Y = \{1, 2, 3\}$ and $X = \{a, b, c, d\}$, $T = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}$, $I = \emptyset$, and let $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, X, \{1\}\}$ defined by $f(a) = f(c) = 1$, $f(b) = f(d) = 2$, the mapping f is feebly – ideal – continuous but not feebly – continuous.

Remark (4-4) :-

The continuous function and feebly – ideal – continuous function are independent.

Example (4-5) :-

Let $X = \{a, b, c\}$ and $T = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ and $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ and let $Y = \{1, 2, 3\}$ and $\sigma = \{\emptyset, X, \{1\}\}$ and defined by $f(c) = 1$, $f(a) = f(b) = 2$ then the mapping f is continuous but not feebly – ideal – continuous.

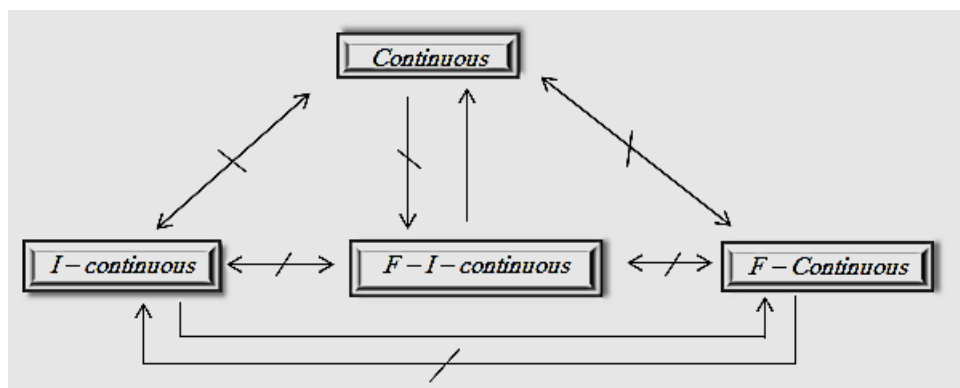
Proposition (4-6):-

Let $f: (X, T, I) \rightarrow (Y, V)$ is ideal continuous function then f is feebly – ideal – continuous function.

Proof:-

Let H is V – open set in Y , and since f is ideal – continuous function, Then $f^{-1}(H)$ is I – open set, and by proposition (3-7) [12] $f^{-1}(H)$ is feebly – ideal – open set in X , from that we get f is feebly – ideal – continuous.

The following diagrams implication gives the relationship among feebly – ideal – continuous , feebly – continuous and ideal – continuous .



Diagrams (4)

Definition (4-7) :-

A function $f: (X, T) \rightarrow (Y, V, J)$ is called feebly – ideal – open if the image of each open set in X is feebly – ideal – open set in Y .

Definition (4-8) :-

A function $f: (X, T, I) \rightarrow (Y, V, J)$ is called feebly – ideal – closed if the image of each closed set in X is feebly – ideal – closed in Y .

Remark (4-9):-

- 1) Open function and feebly – ideal – open function are independent.
- 2) Feebly – open function and feebly – ideal – open function are independent.

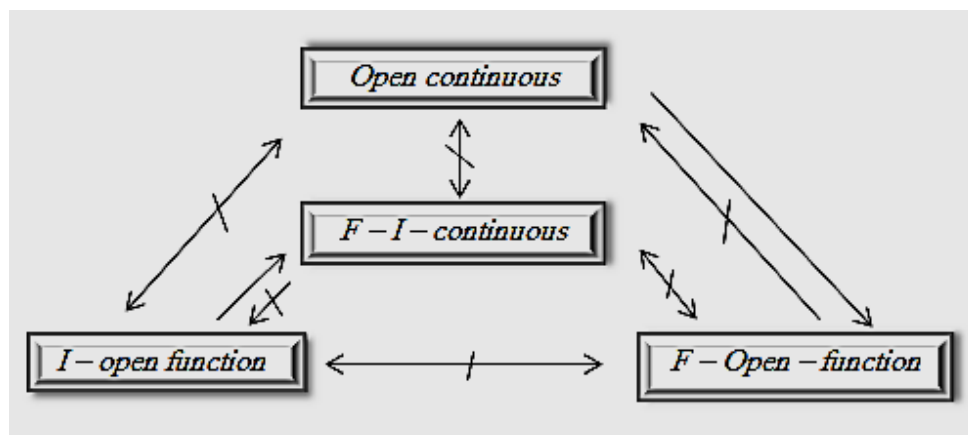
Proposition (4-10):-

If $f: (X, T, I) \rightarrow (Y, V, J)$ is ideal – open function then f is feebly – ideal – open function .

Proof:-

Let E is V – open set in X , and since f is ideal – open function, then $f(E)$ is ideal – open set in X , and by proposition (3-7) [12] $f(H)$ is feebly – ideal – open Set in Y , from that we get f is feebly – ideal – open set .

The following diagrams implication gives the relationship among feebly – ideal – open function, feebly – open function and ideal – open function.



Diagrams (5)

Theorem (4-11):-

Let $f: (X, T, I) \rightarrow (Y, \sigma)$ injective $f-I$ -continuous function and Y is T_2 -space then X is $F-I-T_2$.

Proof:-

Let x, y are two distinct point in X since f is injective, then $f(x) \neq f(y)$ in Y , since Y is T_2 -space then there exist two disjoint U, V are σ -open sets in Y , such that $f(x) \in U, f(y) \in V$ and since f is $F-I$ -continuous, then $f^{-1}(U), f^{-1}(V)$ are disjoint $F-I$ -open sets in X , such that $x \in f^{-1}(U), y \in f^{-1}(V)$ and hence X is $F-I-T_2$ space.

Theorem (4-12):-

If $f: (X, T) \rightarrow (Y, V, I)$ is bijective $F-I$ -open function and X is T_2 -space then Y is $F-I-T_2$ space.

Proof:-

Let y_1, y_2 are two distinct point in Y , then since f is surjective there exist x_1, x_2 in X , such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$ and x_1, x_2 are two distinct point, since X is T_2 -space, then there exist disjoint T -open sets U, V such that $x_1 \in U, x_2 \in V$ and since f is $F-I$ -

open function Then $f(U), f(V)$ are two $F-I$ -open sets $f(U) \cap f(V) = \emptyset$, $f(X_1) \in f(U)$, $f(X_2) \in f(V)$ so Y is $f-I-T_2$ space.

Theorem (4-13):-

If $f: (X, T) \rightarrow (Y, V, I)$ is continuous bijective and $F-I$ -open function and X is regular the Y is $F-I$ -regular.

Proof:-

Let y is any point in Y , H is V -closed set, such that $y \notin H$ since f is surjective continuous function then there exist x in X such that $y = f(x)$ and $f^{-1}(H)$ is closed set in X , and $x \notin f^{-1}(H)$ since X is regular, there two disjoint open sets U, V in X such that $f^{-1}(y) = x \in U$, $f^{-1}(H) \subset V$ since f is $F-I$ -open function $f(U), f(V)$ are two $F-I$ -open set in Y , such that $y \in f(U)$, $H \subset f(V)$ and $f(U) \cap f(V) = \emptyset$, hence Y is $F-I$ -regular.

Theorem (4-14):-

Let $f: (X, T, I) \rightarrow (Y, \sigma)$ closed function and $F-I$ -continuous and Y is regular space then X is $F-I$ -regular.

Proof:-

Let $x \in X$, H is closed set in X such that $x \notin H$, since f is closed function, then $f(x) \in Y$, $f(x) \notin f(H)$, $f(H)$ is closed set in Y since Y is regular, then there exist two disjoint open sets U, V such that $f(x) \in U$, $f(H) \subset V$, and since f is $F-I$ -continuous then $f^{-1}(U), f^{-1}(V)$ are two disjoint $F-I$ -open set in X , $x \in f^{-1}(U)$, $H \subset f^{-1}(V)$ hence X is $F-I$ -regular.

Theorem (4-15):-

Let $f: (X, T, I) \rightarrow (Y, \sigma)$ closed function, $F-I$ -continuous and Y is normal then X is $F-I$ -normal.

Proof:-

Let H_1, H_2 are two disjoint closed set in X , since f is closed function then $f(H_1), f(H_2)$ are two closed disjoint sets in Y , since Y is normal, then there exist two disjoint U -open sets in

Y , such that $f(H_1) \subset V, f(H_2) \subset U$, and since f is $F-I$ -continuous, then there exist $f^{-1}(U), f^{-1}(V)$ are two disjoint $F-I$ -open sets in X , such that $H_1 \subset f^{-1}(V), H_2 \subset f^{-1}(U)$, and so X is $F-I$ -normal.

Theorem (4-16) :-

Let $f: (X, T, I) \rightarrow (Y, V, I)$ continuous bijective $F-I$ -open function and X is normal then Y is $F-I$ -normal.

Proof:-

Let H_1, H_2 are two disjoint closed sets in Y , since f is continuous, then $f^{-1}(H_1), f^{-1}(H_2)$ are disjoint closed sets in X , and since X is normal space then there exist disjoint U, V are T -open sets, such that $f^{-1}(H_1) \subset U, f^{-1}(H_2) \subset V$, and since f is $F-I$ -open function, then there exist $f(U), f(V)$ are $F-I$ -open set in Y , such that $H_1 \subset f(U), H_2 \subset f(V), f(U) \cap f(V) = \emptyset$ and hence Y is $F-I$ -normal.

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