

Using Numerical Methods To Solve Continuous Triple Integrals

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Abstract

In this research we have worked on using numerical methods to solve the continuous, triple integrals that cannot be solved using the conventional methods. The numerical methods which we adopted are applied on the three dimensional so that ($h_1 = h_2 = h_3$), meaning that the number of the partial periods is equal on all the internal, the middle, and the external dimensions for the triple integral periods. We have called these methods (3M) and 2M (S), where (3M) represents the (midpoint rule) which is also applied on three dimensions. The symbol 2M(S) represents the (midpoint rule) on the external dimension z and the middle dimension y , and the Simpson's rule on the internal dimension x . The results were accelerated using the accelerated method (Aitken's Acceleration) to get better results.

Keywords : Triple Integrals ; midpoint rule; Simpson's rule ; Aitken's Acceleration.

1- Introduction

Numerical integration is defined as a study of how to find the approximate value of a particular integration. The application of numerical integration in its beginnings started due to the attempt to find the area of a circle using the Greek quadrature by dividing the circle into regular polygons, thus enabling Archimedes to find the higher and lower limits of the fixed ratio π .

The triple integrals are of great importance such as finding the areas, finding the area within $x^2 + y^2 = 4z$, above $z = 0$, and below $x^2 + y^2 = 4z$, the area within a cylinder $\rho = 4\cos(\theta)$ that is defined from above by a ball $p^2 + z^2 = 16$ and from below by a level $z = 0$ and calculating the middle positions of the area under $z^2 = xy$ and above the rectangular $y = x, y = 0, x = 4$ as well as calculating the determination in inertia of the area within $x^2 + y^2 = 9$ above $z = 0$ and under the level $x + y = 4$. The triple integrals are vividly important in finding the masses of variable density such as cutting a thin wire or a metal sheet (Frank Izer, 6).

One of the researchers who highlighted the value of triple integration in 2010 was Eghaar. [2] A method was introduced to calculate the values of triple integration, a complex method of the Romberg's Acceleration Method and the midpoint rule applied to the three dimensions (internal, mid and external) When the distances between the coordinates on the axis X are equal to the distances between the coordinates on the axis Y and equal to the distances between the coordinates on the axis Z (She named it RMMM). This method proved its advantage in terms of accuracy, number of partial periods used and time taken. The researcher Hussein.S.R in 2015, which applied the trapezoidal rule on all dimensions, internal x , middle y , and external z , when n_1 (The number of partial periods which the period $[a,b]$ is divided into) does not equal n_2 (The number of partial periods which the period $[c,d]$ is divided into) which does not equal n_3 also (The number of partial periods which the period $[e,g]$ is divided into). This means $h_1 \neq h_2 \neq h_3$ if

$h_1 = \frac{b-a}{n_1}$, $h_2 = \frac{d-c}{n_2}$, $h_3 = \frac{g-e}{n_3}$ and the results were enhanced using Romberg's Acceleration method. (Hussein S. R. [3])

2- Calculating the Triple Integrals of Continuous Integrators Using $Ait(3M)$ and $Ait(2M(S))$

2-1 The Ait (3M) Method

This method's formula can be obtained by applying the middle point rule on the three dimensions when $n_1 = n_2 = n_3$ where (n_1 is the number of the partial periods $[a,b]$ period is divided into, and n_2 is the number of the partial periods $[c,d]$ period is divided into, and n_3 is the number of the partial periods $[e,f]$ period is divided into) meaning that $h_1 = h_2 = h_3$, where

$h_1 = \frac{b-a}{n_1}$, $h_2 = \frac{d-c}{n_2}$, $h_3 = \frac{f-e}{n_3}$, let's suppose

$$(1) \dots \text{Int1} = \int_e^f \int_c^d \int_a^b g(x, y, z) dx dy dz$$

since $h_1 = h_2 = h_3$, the numerical format of the integral defined by format (1) can be written using the middle point rule on the three dimensions x, z, y and according to the format:

$$(2) \dots \text{Int1} = \int_e^f \int_c^d \int_a^b g(x, y, z) dx dy dz \approx (3M)(h) = h^3 \sum_{m=1}^n \sum_{l=1}^n \sum_{k=1}^n f(x_k, y_l, z_m)$$

Eghaar B.H. [2]

The (x, y, z) integral is continuous within the integral's area, and

$$x_k = a + \frac{2k-1}{2}h_1, \quad k=1,2,3,\dots,n$$

$$y_l = c + \frac{2l-1}{2}h_2, \quad l=1,2,3,\dots,n$$

$$z_m = e + \frac{2m-1}{2}h_3, \quad m=1,2,3,\dots,n$$

(We have used n to represents the number of the partial periods were $n=n_1=n_2=n_3=n_4$)

When placing ($n_1 = n_2 = n_3 = 1$) in the above formula, the approximate value of the triple integral is calculated through calculating $(3M)(h) = h^3 g(x_1, y_1, z_1)$, fixing in our tables this approximate value when $n_1 = n_2 = n_3 = 1$ then putting $n_1 = n_2 = n_3 = 2$, calculating $(3M)(h) = h^3 \sum_{l=1}^2 \sum_{k=1}^2 \sum_{j=1}^2 g(x_j, y_k, z_l)$, placing this value in the tables. Then, we put $n_1 = n_2 = n_3 = 3$ then calculate $(3M)(h) = h^3 \sum_{l=1}^3 \sum_{k=1}^3 \sum_{j=1}^3 g(x_j, y_k, z_l)$, fixing the value in the tables as well as the approximate value of the triple integral $Int1$.

And we continue to apply the (3M) rule for the rest of the values, then apply the Aitken's acceleration method to accelerate the convergence of values into the real value of the integral, until we get the value according to the needed accuracy.

In order to enhance the convergence, we apply the Aitken's acceleration method on three on every three values that result from formula (2) according to the formula:

$$(3) \dots Ait(3M) \equiv \frac{((3M)(h)) \times \left((3M)\left(\frac{h}{3}\right) \right) - \left((3M)\left(\frac{h}{2}\right) \right)^2}{\left((3M)\left(\frac{h}{3}\right) \right) - 2 \left((3M)\left(\frac{h}{2}\right) \right) + ((3M)(h))}$$

Where $Ait(3M)$ is a value in the new column of the Aitken table, and each of $(3M)(h), (3M)\left(\frac{h}{2}\right), (3M)\left(\frac{h}{3}\right)$ values are in the previous column of the table.

2-2 The $Ait(2M(S))$ Method

Let's assume that the integration $Int2$ is defined as follows:

$$Int2 = \int_e^f \int_c^d \int_a^b g(x, y, z) dx dy dz \quad \dots(4)$$

Where $g(x, y, z)$ is a continuous integral in all of the integration points $[a, b] \times [c, d] \times [e, f]$.

Generally speaking, the Int2 can be written as follows:

$$Int2 = \int_e^f \int_c^d \int_a^b g(x, y, z) dx dy dz = (3M(S))(h_1, h_2, h_3) \quad \dots(5)$$

$(2M(S))$ is the numerical value of the integral using the middle point rule on the y and z axes, and the Simpson rule on the internal axis x , assuming that $\frac{b-a}{n_1} = \frac{d-c}{n_2} = \frac{f-e}{n_3}$ i.e. $h_1 = h_2 = h_3$, and $n_1 = n_2 = n_3$.

Since , then the numerical format of the integral can be written as

$$(3M)(h_1, h_2, h_3) = (3M)(h) = \frac{h^3}{3} \sum_{m=1}^n \sum_{l=1}^n g(a, y_l, z_m) + g(b, y_l, z_m) + 4 \sum_{k=1}^{\frac{m}{2}} g(x_{(2k-1)}, y_l, z_m) + 2 \sum_{k=1}^{\frac{m}{2}-1} g(x_{2k}, y_l, z_m)$$

$$x_{(2k-1)} = a + (2k-1)h, \quad k=1, 2, 3, \dots, \frac{n}{2}$$

$$x_{2k} = a + (2k)h, \quad k=1, 2, 3, \dots, \frac{n}{2}-1$$

$$y_l = c + \frac{2l-1}{2}h, \quad l=1, 2, 3, \dots, n$$

$$z_m = e + \frac{2m-1}{2}h, \quad m=1, 2, 3, \dots, n$$

In order to numerically calculate the triple integrals using the Ait $(2M(S))$, we begin by placing $n=2$ and calculate the approximate triple integral using the $(2M(S))$ which equals:

$$Int2 = \int_e^f \int_c^d \int_a^b g(x, y, z) dx dy dz = \frac{h^3}{3} [g(a, y_1, z_1) + g(a, y_2, z_1) + g(a, y_1, z_2) + g(a, y_2, z_2) + g(x_2, y_1, z_1) + g(x_2, y_2, z_1) + g(x_2, y_1, z_2) + g(x_2, y_2, z_2) + 4(g(x_1, y_1, z_1) + g(x_1, y_2, z_1) + g(x_1, y_1, z_2) + g(x_1, y_2, z_2))] \quad \dots(6)$$

fixing in our tables the approximate value when $n=2$, then placing $n=4$ and we calculate the integral value.

$$Int2 = \int_e^f \int_c^d \int_a^b g(x, y, z) dx dy dz = \frac{h^3}{3} \sum_{m=1}^4 \sum_{l=1}^4 [g(a, y_l, z_m) + g(b, y_l, z_m) + 4 \sum_{k=1}^2 g(x_{(2k-1)}, y_l, z_m) + 2g(x_{2k}, y_l, z_m)] \quad \dots(7)$$

We also fix this value in our tables as the approximate value of the triple integral and calculate the value of the rule when $n=8$

$$Int2 = \int_e^f \int_c^d \int_a^b g(x, y, z) dx dy dz = \frac{h^3}{3} \sum_{m=1}^8 \sum_{l=1}^8 [g(a, y_l, z_m) + g(b, y_l, z_m) + 4 \sum_{k=1}^4 g(x_{(2k-1)}, y_l, z_m) + 2 \sum_{k=1}^3 g(x_{2k}, y_l, z_m)] \quad \dots(8)$$

$$x_{(2k-1)} = a + (2k-1)h, \quad k=1, 2, 3, 4$$

$$x_{2k} = a + (2k)h, \quad k=1, 2, 3$$

$$y_l = c + \frac{2l-1}{2}h_2, \quad l=1, 2, 3, \dots, 8$$

$$z_m = e + \frac{2m-1}{2}h_3, \quad m=1, 2, 3, \dots, 8$$

We fix the value in our tables. In order to enhance the approximate value, we apply the Aitken acceleration on every three values that result from the table using the formula:

$$Ait(2M(S)) \cong \frac{((2M(S))(h)) \times \left((2M(S))\left(\frac{h}{3}\right) \right) - \left((2M(S))\left(\frac{h}{2}\right) \right)^2}{\left((2M(S))\left(\frac{h}{3}\right) \right) - 2 \left((2M(S))\left(\frac{h}{2}\right) \right) + ((2M(S))(h))} \quad \dots(9)$$

Where $Ait(2M(S))$ is a value in the new column of the Aitken table and so is each of $(2M(S))(h), (2M(S))\left(\frac{h}{2}\right), (2M(S))\left(\frac{h}{3}\right)$.

This process is merely accelerating the convergence of resulting values in some method towards the final value $Ait(3M)$ and $Ait(2M(S))$. If we, for instance, had the n value using the $(3M)$ or $(2M(S))$, we will get $n-2$ value using the Aitken acceleration method, using the no.9 formula, or no.3 formula, then we can also use the Aitken acceleration method on $n-2$ to get $n-4$ and so on, until we get the needed accuracy.

Note: (1) Using the Ait(3) method, we used the values $n = 1, 2, 4, 8, 16, \dots$ Nasir [7]

(2) Using the Ait(2M(S)) method, we used the values $n = 1, 2, 3, 8, 16, \dots$ because the Simpson rule needs an even number of partial periods, and that does not affect the middle point rule. S.S.Sastry [5].

3- Examples

$$Int = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} x \cos(3y - 2z) dx dy dz$$

The integrator of the integration

is defined for each of the

$$(x, y, z) \in \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$$

since its real value is 0.411233517 (close to nine decimal

Ait(3M) and Ait(2M(S))

places), and when applying the two methods

we got the following

results:

n	(3M)	Ait(3M)	Ait(3M)
1	2.152457151		
2	0.582450708		
4	0.447425678	0.434720455	
8	0.419932710	0.412903521	
16	0.413387368	0.411342196	0.411221847
32	0.411770683	0.411240383	0.411233280
64	0.411367728	0.411233947	0.411233513

Table (1): Calculating triple integrals $Int = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} x \cos(3y - 2z) dx dy dz$ **using Ait (3M)**

n	(2M(S))	Ait(2M(S))	Ait(2M(S))
2	0.582450708		
4	0.447425678		
8	0.419932710	0.412903521	
16	0.413387368	0.411342196	
32	0.411770683	0.411240383	0.411233280
64	0.411367728	0.411233947	0.411233513

Table (2): Calculating triple integrals $Int = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} x \cos(3y - 2z) dx dy dz$ **using the Ait (2M(S))**

From table (1), we conclude that both methods results in the same outcomes. When $n=64$, the integration value using the 3M and the 2M(S) is correct close to only three decimal places. However, when using the Aitken acceleration methods, we got correct results close to 8 decimal places when $n=64$ using both of Ait (3M) and Ait (2M(S)).

Also the integrator of the integration is defined for all since its true value is 5.073214112 (close to nine decimal places) and the results that we have got using the (3M) method and the (2M(S)) method as well as the Aitken acceleration are mentioned in the tables (3) and (4).

n	(3M)	Ait(3M)	Ait(3M)
1	4.481689070		
2	4.917446594		
4	5.033754508	5.076100877	
8	5.063316448	5.073390786	
16	5.070737642	5.073225097	5.073214308
32	5.072594866	5.073214797	5.073214115
64	5.073059292	5.073214155	5.073214112

Table (3): Calculating the triple integration $Int = \int_0^1 \int_0^1 \int_0^1 e^{(x+y+z)} dx dy dz$ using the Ait (3M)

n	(2M(S))	Ait(2M(S))	Ait(2M(S))
2	4.970505562		
4	5.046982204		
8	5.066620381	5.073405540	
16	5.071563427	5.073226135	
32	5.072801300	5.073214864	5.073214109
64	5.073110900	5.073214159	5.073214112

Table (4): Calculating the triple integral $Int = \int_0^1 \int_0^1 \int_0^1 e^{(x+y+z)} dx dy dz$ using the Ait (2M(S))

From both tables, we conclude that when $n_1 = n_2 = n_3 = n_4 = 64$, the value we have got from the integral using the (3M) and the (2M(S)) methods are correct for only three decimal places, but when using the Aitken acceleration method separately, we reached a result that is correct for nine decimal places when $n=64$ (2^{18} partial periods), meaning that it matched the analytical values (close to nine decimal places).

However, the integrator of the integration is continuous for every , since its real value is 0.0167244986 (close to nine decimal places) and the results are mentioned in the tables in both of the Ait (3M) and Ait (2M(S)) methods.

n	(3M)	Ait(3M)	Ait(3M)
1	0.0165289256		
2	0.0166746700		
4	0.0167119800	0.0167248176	
8	0.0167213651	0.0167245193	
16	0.0167237150	0.0167244999	0.0167244986

Table (5): Calculating the triple integral $Int = \int_{3.5}^4 \int_{2.5}^3 \int_{1.5}^2 \frac{y}{\sqrt{(1+xz)^3}} dx dy dz$ using the Ait (3M)

n	(2M(S))	Ait(2M(S))	Ait(2M(S))
2	0.0167170571		
4	0.0167223931		
8	0.0167239559	0.0167246034	
16	0.0167243619	0.0167245044	
32	0.0167244644	0.0167244990	0.0167244986

Table (6): Calculating the triple integral $Int = \int_{3.5}^4 \int_{2.5}^3 \int_{1.5}^2 \frac{y}{\sqrt{(1+xz)^3}} dx dy dz$ using the Ait (2M(S))

We note from table (5) that when the value of the integration using the 3M method is correct method of five decimal places and using the Aitken acceleration with it, we reached a value corresponding to the analytical value close to ten decimal places. When $n=16$ (2^{12} partial period) the convergence of the values obtained was quick towards the analytical value in a relatively small number of partial periods use.

The table (6), however, the integration value using the (2M(S)) when $n=16$ is correct for five decimal places, and for seven decimal places when $n=32$. The results were then accelerated using the Ait(2M(S)) where we have got a match with the analytical value close to ten decimal when $n=32$ (2^{15} partial periods).

4- Conclusion

We conclude from the above tables that when calculating the triple integrals using the Ait (2M(S)) and the Ait(3M) for the axes x, y, z and when $n_1 = n_2 = n_3$ and ($h_1 = h_2 = h_3$), that the (2M(S)) and (3M) give correct values (for several decimal places or digits) when compared to

the actual values of the integrations using a number of partial periods without using the Aitken acceleration on both of (2M(S)) and (3M) methods, the results were better in terms of the approaching speed with a few number of partial periods in relation to the real values of integrations, and that the behavior of the integrator had a great effect in the speed at which in the results approaching the analytical values and the used partial periods.

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