On the normed BH-algebras

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Abstract:
The main purpose of this paper, The notions of norm and distance in BH-algebras are introduced, and some basic properties in normed BH-algebras are given. It is obtained that the isomorphic (homomorphic) image and inverse image of a normed BH-algebra are still normed BH-algebras. The relations of normed properties between BH-algebra and Cartesian product of BH-algebras are investigated. The limit notion of sequence of points in normed BH-algebras is introduced, and its related properties are investigated, the relations between bounded normed BH-algebras and fuzzy BH-algebras are discussed.

Key words: BH-algebra, norm, normed BH-algebra, limit, fuzzy subalgebra.

Subject Classification: 06F35; 03G25; 94D05.

Introduction:

1. Preliminaries
In this section, we give some basic concepts about a BH-algebra, associative BH-algebra and fuzzy subalgebra to BH-algebra.

Definition(1.1):[5]A BH-algebra is a nonempty set X with a constant 0 and a binary operation "*" satisfying the following conditions:
   i. \( x*x=0 \), for all \( x \in X \).
   ii. \( x*y=0 \) and \( y*x=0 \) imply \( x=y \), for all \( x, y \in X \).
   iii. \( x*0=x \), for all \( x \in X \).
   iv. 

Remark(1.2):[11]
1. Every BCK-algebra is a BH-algebra.
2. Every BCH-algebra is a BH-algebra.
3. Every BCI-algebra is a BH-algebra.

Remark (1.3): Let X be a BH-algebra, the order relation "≤" is defined on X by x≤y if and only if x*y=0.

Definition (1.4): A BH-algebra X is called an associative BH-algebra if (x*y)*z=x*(y*z), for all x,y,z∈X.

Proposition (1.5): Let X be an associative BH-algebra. Then the following properties hold:
   i. 0*x=x, for all x∈X.
   ii. x*y=y*x, for all x∈X.

Definition (1.6): Let X be a BH-algebra. Then the set X⁺={x∈X :0*x=0} is called the BCA-part of X.

Remark (1.7): Let X and Y be BH-algebras. A mapping f(X,*,0)→(Y,*,0') is called a homomorphism if f(x*y)=f(x)*f(y) for any x,y∈X. A homomorphism f is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two BH-algebras X and Y are said to be isomorphic, written X≅Y, if there exists an isomorphism f:X→Y. For any homomorphism f:X→Y, the set {x∈X :f(x)=0'} is called the kernel of f, denoted by Ker(f), the set {f(x):x∈X} is called the image of f, denoted by Im(f), and these {x∈X :f(x)=y, for some y∈Y} is a preimage of f, denoted by f⁻¹(Y). Notice that f(0)=0'.

Definition (1.8): Let A be a fuzzy set in X, for all t∈[0,1]. The set Aₜ={x∈X :A(x)≥t} is called a level subset of A.

Definition (1.9): A fuzzy set A in a BH-algebra X is said to be a fuzzy subalgebra of X if it satisfies: A(x*y)≥min {A(x),A(y)}, for all x,y∈X.

Definition (1.10): A map x→∥x∥ and E into R is a norm on E if and only if
   1. ∥x∥≥0 and ∥x∥=0 if x=0.
   2. ∥αx∥=|α|∥x∥.
   3. ∥x+y∥≤∥x∥+∥y∥ ∀ x,y∈E.

2. The main results: In this section, we find the normed BH-algebra, The Limits in Normed BH-algebra, Bounded Normed BH-algebras and Fuzzy Subalgebras of BH-algebras.

(2.1) Let R be a non-negative real numbers set. A BH-algebra X is called a normed BH-algebra if there is a mapping p:X→R satisfying: ∀x,y,z∈X,
   (i) p(x)=0 ⇔ x=0;
   (ii) p(x*y)≤p(x)+p(z)+p(z*y);
where p(x) is the norm of x, and is denoted by ∥x∥.

Definition (2.2) Let X be a normed BH-algebra, and x, y∈X. A norm ∥x+y∥ is said to be a distance of x and y, and is denoted by d(x, y).

By Definition(2.1), Definition (2.2) and Definition (1.1), we have
(iii) \(d(x, 0)\geq 0\), and \(d(x, 0) = 0 \iff x = 0\);
(iv) \(d(x, y) \leq d(x, z) + d(z, y)\); for all \(x, y, z \in X\).

**Example (2.3)** Let \(X\) be an arbitrary non-empty set, and \(P(X)\) be the power set of \(X\). Then \((P(X); \cap, \emptyset)\) is a BCI-algebra then \((P(X); \cap, \emptyset)\) is a BH-algebra. Define \(\|A\| = |A|\) for all \(A \in P(X)\), where \(|A|\) is the potential of \(A\), then \(P(X)\) is a normed BH-algebra. In fact, (i) is obvious. Since \(\forall A, B, C \in P(X), A \setminus B \subseteq (A \setminus C) \cup (C \setminus B), |A \setminus B|\leq |A \setminus C| + |C \setminus B|\), that is, \(\|A \setminus B\|\leq \|A \setminus C\| + \|C \setminus B\|\), therefore, (ii) holds.

**Theorem (2.4)** Let \(X\) be a normed BH-algebra. For any \(x, y, z \in X\), we have
1. \(d(0, x) = d(x, 0) = 0\) and \(d(x, 0) = \|x\|\).
2. \(d(x, 0) = 0\) implies \(x = 0\).
3. \(d(x, y) = d(y, x) = 0\) implies \(x = y\).
4. \(x \leq y\) implies \(d(x, z) \leq d(y, z)\) and \(d(z, y) \leq d(z, x)\).

**Proof:** Using the concept, properties and Definition of BH-algebra, we can prove that the following results are true.

**Theorem (2.5):** Let \(f\) be an isomorphic mapping from BH-algebra \(X\) to BH-algebra \(X'\). If \(X\) is a normed BH-algebra, then \(X'\) is also a normed BH-algebra.

**Proof** For any \(x' \in X'\), define \(\|x'\| = \|f^{-1}(x')\|\), then \(X'\) is a normed BH-algebra. In fact, since \(f\) is an isomorphic mapping from BH-algebra \(X\) to BH-algebra \(X'\), for any \(x \in X\), there exists \(x' \in X'\) such that \(f(x) = x'\). Therefore, \(\|x'\| = \|x\|\geq 0\). But \(\|x\| = 0 \iff \|f^{-1}(x')\| = 0 \iff \|f^{-1}(x') = 0 \iff x = f(0) = 0\). Therefore, (i) holds. For \(x', y', z' \in X'\), there exist respectively unique \(x, y, z \in X\), such that \(f(x) = x', f(y) = y'\) and \(f(z) = z'\). Therefore,

\[
\|x' * y'\| = \|f^{-1}(x) * f^{-1}(y)\|
\]

\[
= \|f^{-1}(x) * f^{-1}(y)\| = \|x + y\|
\]

\[
\leq \|x + z\| + \|z + y\|
\]

\[
= \|f^{-1}(x') * f^{-1}(z')\| + \|f^{-1}(z') * f^{-1}(y')\|
\]

\[
= \|f^{-1}(x') * z'\| + \|f^{-1}(z') * y'\|
\]

\[
= \|x' * z'\| + \|z' * y'\|.
\]

Thus (ii) holds and the proof is completed.

**Theorem (2.6):** Let \(f\) be a homomorphism from finite BH-algebra \(X\) onto BH-algebra \(X'\). If \(X\) is a normed BH-algebra, then \(X'\) is also a normed BH-algebra.

**Proof** For any \(x' \in X'\), define \(\|x'\| = \inf_{f^{-1}(x')} \|x\|\). Then \(X'\) is a normed BH-algebra. In fact, since \(f\) is epimorphism, \(f^{-1}(x') \neq \emptyset\) for any \(x' \in X'\), so \(\|x'\| = \inf_{f^{-1}(x')} \|x\|\geq 0\). If \(\|x\| = 0\), owing to \(X\) being finite, there exists \(x \in f^{-1}(x')\) such that \(\|x\| = 0\). Since \(X\) is a normed BH-algebra, \(x = 0\), and so \(x' = f(x) = f(0) = 0\). Conversely, if \(x' = 0\), then \(\|x'\| = \inf_{f^{-1}(x')} \|x\| = |0| = 0\) since \(0 \in f^{-1}(0)\). This proves (i). For any \(x', y', z' \in X'\), there exist \(x, y, z \in X\), such that \(f(x) = x', f(y) = y'\) and \(f(z) = z'\). Therefore,

\[
\|x'y'\| = \inf_{f^{-1}(x'y')} \|w\| = \inf_{f^{-1}(x'y')} \inf_{f^{-1}(y')} \|x\| + \inf_{f^{-1}(x'y')} \|x'y\| = \inf_{f^{-1}(x'y')} \|x\| + \inf_{f^{-1}(x'y')} \|x'y\|
\]

From \(\|x'y\| \leq \|x\|z\| + \|z\|y\|\), we have
Proof For any x in X, define ||x|| = ||f(x)||, then X is a normed BH-algebra. In fact, because X’ is a normed BH-algebra, we have ||x|| = ||f(x)|| ≥ 0. Thus ||x|| = 0 ⇔ ||f(x)|| = 0 ⇔ f(x) = 0 ⇔ x = 0. This proves (i). For any x, y, z in X, we have.

\[ ||x * y|| ≤ ||f(x)|| * ||f(y)|| ≤ ||f(x)|| * (||f(y)|| + ||f(z)||) + ||f(z)|| * ||f(y)|| = ||f(x)|| * ||z|| + ||f(z)|| * ||y||.\]

Therefore, (ii) holds and X is a normed BH-algebra.

Remark (2.8): Let \( X_i \) (1 = 1, 2, \cdots, n) be a BH-algebra and a mapping \( \rho_i : X \to X_i \) be a projection from the Cartesian product \( X = X_1 \times X_2 \times \cdots \times X_n \) to the i coordinate set \( X_i (1 = 1, 2, \cdots, n) \). Then for any \( x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \) in X, we have

\[ \rho_i(x * y) = \rho_i(x_1 * y_1, x_2 * y_2, \cdots, x_n * y_n) = x_1 * y_1 = \rho_i(x_1, y_1). \]

That is, the projection \( \rho_i \) is an epimorphism from X to \( X_i \).

Theorem (2.9): Let \{X_i : i = 1, 2, \cdots, n\} be a finite family of BH-algebras and X be the product algebra. Then X is a normed BH-algebra if and only if X is a normed BH-algebra for each i = 1, 2, \cdots, n.

Proof: Since \( \rho_i : X \to X_i \) is an epimorphism from X to \( X_i \), \( X_i \) is a normed BH-algebra for each i = 1, 2, \cdots, n by Theorem (2.6).

\[ ||x \cdot y|| = ||(x_1 \cdot 1, x_2 \cdot 1) * (y_1 \cdot 1, y_2 \cdot 1)|| = ||(x_1 * y_1, x_2 * y_2)|| ≤ ||x_1 * y_1|| + ||x_2 * y_2|| \]

This proves (ii) and X is a normed BH-algebra.
**Theorem (2.11):** In a normed BH-algebra, the limit of any convergence sequence is unique.

**Proof:** Suppose $x_n \to x_0$ and $x_n \to y_0$. By Definition (2.10), $\forall \varepsilon > 0$, there exists a positive integer $N$, such that $d(x_n, x_0) < \frac{\varepsilon}{2}$, $d(y_0, y_m) < \frac{\varepsilon}{2}$ and $d(x_n, y_0) < \frac{\varepsilon}{2}$ for $n > N$.

So $d(x_n, y_0) \leq d(x_n, x_0) + d(x_0, y_0) < \varepsilon$. Similarly, $d(y_0, x_0) < \varepsilon$. By arbitrariness of $\varepsilon$, we get $d(x_n, y_0) = d(y_0, x_0)$, and hence $x_0 = y_0$.

**Theorem (2.12):** In a normed BH-algebra, the necessary condition that a point sequence $\{x_n\}$ is convergent, then for any $\varepsilon > 0$, there exists $N$, such that $d(x_m, x_n) < \varepsilon$ for $m, n > N$.

**Proof:** Let $x_n \to x_0$. From Definition (2.10), for any $\varepsilon > 0$, there exists $N$, such that $d(x_n, x_0) < \frac{\varepsilon}{2}$ and $d(x_m, x_n) < \frac{\varepsilon}{2}$ for $m > N$. So for $m > N$, we have also that $d(x_m, x_0) < \frac{\varepsilon}{2}$ and $d(x_m, x_n) < \frac{\varepsilon}{2}$. Thus for $n, m > N$, we get that $d(x_n, x_m) \leq d(x_m, x_0) + d(x_0, x_n) < \varepsilon$.

**Theorem (2.13):** In a normed BH-algebra, the point sequence $\{x_n\}$ is convergent if and only if any non-trivial subsequence of $\{x_n\}$ is convergent.

**Proof:** Necessity. Suppose $\{x_{n_k}\}$ is an arbitrary non-trivial subsequence of $\{x_n\}$, and $x_n \to x_0$. Then $\forall \varepsilon > 0$, there exists $N$, such that $d(x_m, x_0) < \varepsilon$ and $d(x_0, x_n) < \varepsilon$ for $n > N$. Taking $K = N$, for $k > K$, we have $n_k \geq k > K = N$. So $d(x_{n_k}, x_0) < \varepsilon$ and $d(x_0, x_{n_k}) < \varepsilon$, i.e., $\lim_{k \to \infty} x_{n_k} = x_0 = \lim_{k \to \infty} x_n$.

Sufficiency. Since $\{x_{2k-1}\}$, $\{x_{2k}\}$ and $\{x_{3k}\}$ are non-trivial subsequences of $\{x_n\}$, we have that these three subsequences are convergent by hypothesis. Obviously, $\{x_{2k}\}$ is a subsequence of $\{x_{2k-1}\}$ and $\{x_{3k}\}$, and therefore the following conclusion holds by the proof of necessary condition $x_{2k} = \lim_{k \to \infty} x_{2k} = \lim_{k \to \infty} x_{3k}$. Hence, $\lim_{k \to \infty} x_{2k} = \lim_{k \to \infty} x_{2k-1}$, and we can check that $\{x_n\}$ is convergent.

**Theorem (2.14):** In a normed BH-algebra $X$, if $\lim_{n \to \infty} x_n = x_0$, then for any $y_0$ in $X$, $\{d(x_n, y_0)\}$ and $\{d(y_0, x_n)\}$ are bounded sequences.

**Proof:** By hypothesis, $\{d(x_n, y_0)\}$ and $\{d(x_n, y_0)\}$ as sequence converging to zero are bounded, and so there exists $M > 0$ such that $d(x_n, x_0) \leq M$ and $d(x_0, x_n) \leq M$. Hence, $d(x_n, y_0) \leq d(x_n, x_0) + d(x_0, y_0) \leq M + d(x_0, y_0)$ and $d(y_0, x_n) \leq d(y_0, x_0) + d(x_0, x_n) \leq M + d(y_0, x_0)$.

Therefore, $\{d(x_n, y_0)\}$ and $\{d(y_0, x_n)\}$ are bounded sequences.

**Theorem (2.15):** In a normed BH-algebra, if $\lim_{n \to \infty} x_n = x_0$, then $\lim_{n \to \infty} (x_n * x_0) = 0$.

**Proof:** By Theorem (2.4) and Definition (2.2), we get $d(x_n * x_0, 0) = ||x_n * x_0|| = d(x_n, x_0)$ and $d(0, x_n * x_0) = 0$, and so we can prove $\lim_{n \to \infty} (x_n * x_0) = 0$. 

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Theorem (2.16): In a normed BH-algebra, if \( \lim_{n \to \infty} x_n = x_0 \) and the point sequence \( \{z_n\} \) satisfies that there exists \( N_0 \), such that \( x_n \leq z_n \leq y_n \) for \( n > N_0 \), then \( \{z_n\} \) is convergent, and \( \lim_{n \to \infty} z_n = x_0 \).

Proof: For \( n > N_0 \), \( x_n \leq z_n \leq y_n \). It follows from Theorem (2.4) that \( d(x_n, x_0) \leq d(z_n, x_0) \leq d(y_n, x_0) \) and \( d(x_n, x_0) \leq d(z_n, x_0) \leq d(y_n, x_0) \). So we can prove that \( \lim_{n \to \infty} z_n = x_0 \).

Theorem (2.17): Let \( f \) be a mapping from normed BH-algebra \( X \) to normed BH-algebra \( X' \). If \( \|x\| = \|f(x)\| \) for any \( x \) in \( X \), then for an arbitrary point sequence \( \{x_n\} \) of \( X \), the following conclusions hold:

1. If \( f \) is homomorphic, then \( \lim_{n \to \infty} x_n = x_0 \) implies \( \lim_{n \to \infty} f(x_n) = f(x_0) \);
2. If \( f \) is isomorphic, then \( \lim_{n \to \infty} x_n = x_0 \) if and only if \( \lim_{n \to \infty} f(x_n) = f(x_0) \).

Proof: (1) By hypothesis, we have \( \|f(x_0)\| = \|f(x_0)\| = \|x_0\| \) and \( \|f(x_0)^*f(y_0)\| = \|f(x_0)^*f(y_0)\| = \|x_0\| \). So we can get that \( \lim_{n \to \infty} f(x_n) = f(x_0) \).

(2) The necessity can be proved from (1). Since \( f \) is isomorphic, then \( f^{-1} \) is also an isomorphic mapping from normed BH-algebra \( X' \) to normed BH-algebra \( X \). By (1), \( \lim_{n \to \infty} f^{-1}(f(x_0)) = \lim_{n \to \infty} f^{-1}(f(x_n)) = f^{-1}(f(x_0)) = x_0 \). This completes the proof.

Theorem (2.18): Let \( X_1, X_2 \) be normed BH-algebras with the distances \( d_1, d_2 \) respectively. For any \( z_1 = (x_1, y_1), z_2 = (y_1, y_2) \in X = X_1 \times X_2 \), define \( d(z_1, z_2) = d_1(x_1, y_1) + d_2(y_1, y_2) \), then

1. \( X = X_1 \times X_2 \) is a product BH-algebra with the distance \( d \);
2. \( \{x_n, y_n\} \) is convergent in \( X \) if and only if \( \{x_n\} \) and \( \{y_n\} \) are respectively convergent in \( X_1 \) and \( X_2 \).

Proof (1) From Theorem (2.4), \( \|x\|_{X_1} = d_1(x, 0) \) and \( \|y\|_{X_2} = d_2(x, 0) \) are norms of \( X_1 \) and \( X_2 \) respectively. From the proof of Theorem (2.9), \( \|z\|_X = \|x\|_{X_1} + \|y\|_{X_2} \) for \( z = (x, y) \in X \) is a norm of \( X \). By Definition (2.2), the distance \( d_X \) corresponding to \( \|z\|_X \) satisfies
\[
d_X(z_1, z_2) = \|z_1 - z_2\|_X
= \|x_1 - x_2, y_1 - y_2\|_X
= \|x_1 - x_2\|_{X_1} + \|y_1 - y_2\|_{X_2}
= d_1(x_1, x_2) + d_2(y_1, y_2)
= d(z_1, z_2).
\]
This proves (1).

(2) Let \( \{w_n\} = \{(x_n, y_n)\}, w_0 = (x_0, y_0) \in X \). Suppose that \( \lim_{n \to \infty} w_n = w_0 \), by hypothesis, we have \( d_1(x_n, x_0) \leq d(w_n, w_0) \) and \( d_2(y_n, y_0) \leq d(w_n, w_0) \). Hence, \( \lim_{n \to \infty} x_n = x_0 \) similarly. \( \lim_{n \to \infty} y_n = y_0 \). Conversely, suppose \( x_n \to x_0 \) and \( y_n \to y_0 \). Using \( d(w_n, w_0) = d_1(x_n, x_0) + d_2(y_n, y_0) \) and \( d(w_0, w_n) = d_1(x_0, x_n) + d_2(y_0, y_n) \), we can prove that \( w_n \to w_0 \).

Remark (2.19): (1) Definition (2.1) indicates that a given norm determines a distance, while Theorem (2.18) shows that a given distance determines a norm.

(2) Under the conditions of Theorem (2.18), \( (x_n, y_n) \to (x_0, y_0) \) if and only if \( x_n \to x_0 \) and \( y_n \to y_0 \). This shows that the limits of two-dimensional normed BH-algebra can be transformed into the limits of one-dimensional normed BH-algebra by using Theorem (2.18). This conclusion can be extended to the finite-dimensional case.

(3) For limits of finite-dimensional normed BH-algebra, we can make some discussions similar to
Theorem (2.11) – Theorem (2.14).

**Definition (2.20):** Let X be a normed BH-algebra, \( x_0 \in X \) and r be a positive real number. The point set \( U(x_0; r) = \{ x \in X | d(x, x_0) < r \} \) is called an open neighborhood of \( x_0 \), where \( x_0 \) and r are center and radius of the neighborhood respectively. Similarly, \( \bar{U}(x_0; r) = \{ x \in X | d(x, x_0) \leq r \} \) is called a closed neighborhood of \( x_0 \) with center \( x_0 \) and radius r.

**Theorem (2.21):** Let \( f \) be an isomorphic mapping from BH-algebra X to BH-algebra \( X' \). If \( \| x \| = \| f(x) \| \) for any \( x \) in X, then for any \( x_0 \) in X, the following conclusions hold:

1. Any neighborhood of \( x_0 \) must be the subalgebra of X;
2. \( x \in U(x_0; r) \iff f(x) \in U(f(x_0); r) \), and hence \( U(x_0; r) \) and \( U(f(x_0); r) \) are isomorphic.

**Proof:**
(1) For any \( x, y \in U(x_0; r) \), by \( x * y \leq x \) and Theorem (2.4), we have \( d(x*y, x_0) \leq d(x, x_0) < r \). So, \( x*y \in U(x_0; r) \). Similarly, \( y * x \in U(x_0; r) \). Therefore, \( U(x_0; r) \) is a subalgebra of X. Similarly, \( \bar{U}(x_0; r) \) is also a subalgebra of X.

(2) \( x \in U(x_0; r) \iff d(x, x_0) < r \iff \| f(x) * f(x_0) \| < r \iff \| f(x) \| < r \iff d(f(x), f(x_0)) < r \iff f(x) \in U(f(x_0); r) \).

**Theorem (2.22):** Let \( X_1, X_2 \) be normed BH-algebras with the distances \( d_1, d_2 \) respectively. For any \( z_1 = (x_1, y_1) \), \( z_2 = (y_2, y_2) \) \( \in X = X_1 \times X_1 \), define \( d(z_1, z_2) = d_1(x_1, y_1) + d_2(x_2, y_2) \). Then for any \( z_0 = (x_0, y_0) \in X \), the following conclusions hold:

1. \( \forall r > 0, U(z_0; r) \) is a subalgebra of X;
2. \( z = (x, y) \in U(z_0; r) \) implies \( x \in U(x_0; r) \) and \( y \in U(y_0; r) \);
3. \( x \in U(x_0; r) \) and \( y \in U(y_0; r) \) imply \( z = (x, y) \in U(z_0; 2r) \).

**Proof:**
(1) From Theorem (2.18), \( X = X_1 \times X_1 \) is a product BH-algebra with the distance d, and so \( U(z_0; r) \) is a subalgebra of X.

(2) \( z = (x, y) \in U(z_0; r) \iff d(z, z_0) < r \iff d_1(x, x_0) + d_2(y, y_0) < r \). Then \( d_1(x, x_0) < r \) and \( d_2(y, y_0) < r \), i.e., \( x \in U(x_0; r) \) and \( y \in U(y_0; r) \).

(3) Similar to the proof of (2), we can obtain the proof of (3).

**Remark (2.23):** In Theorem (2.21) and Theorem (2.22), the conclusions about the open neighborhood hold also for the corresponding closed neighborhood.

**Definition (2.24):** Let X be a bounded normed BH-algebra. X is called a bounded normed BH-algebra, if there exists \( M > 0 \) such that \( \| x \| \leq M \) for any \( x \) in X.

By Definition (2.24), if X is a bounded normed BH-algebra, then \( \inf_{x \in X} \{ M \| x \| \leq M \} \) must exist uniquely, we denote it by \( \inf X \). Combining Theorem (2.21) and Theorem (2.22), we have the following conclusions.

**Theorem (2.25):** Under the conditions of Theorem (2.21), X is a bounded normed BH-algebra if and only if \( X' \) is a bounded normed BH-algebra.

**Theorem (2.26):** Under the conditions of Theorem (2.22), X is a bounded normed BH-algebra if and only if both \( X_1 \) and \( X_2 \) are bounded normed BH-algebras.

**Theorem (2.27):** Let X be a bounded normed BH-algebra with the maximal element 1 and \( \| \| \), a norm
of X. Then X is a bounded normed BH-algebra, and inf X = ∥1∥.

**Proof:** For any x in X, we have x ≤1. From Theorem (2.4), ∥x∥ ≤ ∥1∥. Thus inf X = ∥1∥ since 1 ∈ X. This completes the proof.

**Theorem (2.28):** Let X be a bounded normed BH-algebra and inf X = m > 0. If the mapping μ : X → [0, 1] satisfies μ(x) = 1 - ∥x∥/m for any x ∈ X, then μ is a fuzzy subalgebra of X.

**Proof:** From 0 ≤ ∥x∥ ≤ m, we have 0 ≤ 1 - ∥x∥/m ≤ 1, and so μ is a fuzzy subset of X. For any x, y ∈ X, we have ∥x*y∥ ≤ ∥x∥ by x*y ≤ x. Therefore,

\[ μ(x*y) = 1 - \frac{∥x*y∥}{m} ≥ 1 - \frac{∥x∥}{m} = μ(x) ≥ \min(μ(x), μ(y)) \]

Thus μ is a fuzzy subalgebra of X.

**Remark (2.29):** In Theorem (2.28), the fuzzy subalgebra μ is generated by the norm ∥·∥ of bounded normed BH-algebra, so μ is called a fuzzy subalgebra induced by the norm.

**Theorem (2.30):** Let X be a bounded normed BH-algebra and μ be a fuzzy subalgebra induced by the norm. Then the set of all closed neighborhoods of 0 is equal to the set of all level subsets of μ.

**Proof:** Let inf X = m > 0. Then we need only to prove \( \bar{U}(0; t) = \frac{μ^{-1}(m)}{m} \) for any \( t ∈ [0, 1] \).

In fact, \( x ∈ \bar{U}(0; t) ⇔ d(x, 0) ≤ t ⇔ ∥x∥ ≤ t ≤ 1 - \frac{∥x∥}{m} ≤ 1 - \frac{1}{m} ⇔ μ(x) ≥ 1 - \frac{1}{m} ⇔ x ∈ \frac{μ^{-1}(m)}{m} \).

**References**


