Power Classes of Composition Operators on the Complex Hilbert Space and $L^2$ space

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Abstract

In this paper we introduce a new class of operators called $n$ – power- $k$- Quasi- $P$-normal operators acting on the Complex Hilbert Space $H$, and it is defined as $(T^*T)^k(T^n + T^*n) = (T^n + T^*n)(T^*T)^k$ for $n > 1$ and $k$ is any fixed positive integer and it is denoted by $[n,k]QPN]$. Some characterizations on $n$ power $k$ – quasi – $P$ – normal composition operators , $n$ - Power $k$ quasi normal operators , $n$ power - $k$ – quasi normal composition operator, $n$ – power - $k$ – quasi normal weighted composition operator, $n$ – power- $k$ – quasi normal composite multiplication operators and characterizations on weighted shift operator are discussed. Inclusion relations among the various classes of normal operators are characterized and showed in the diagram.

Mathematics Subject Classification: 47B33

Keywords: Quasi normal operator, $k$ – Quasi normal operator, Composition operator, $n$- Power Quasi normal composition operator and composite multiplication operator.

1.0 Introduction:

Let $H$ be a Hilbert space. Let $L(H)$ be the algebra of all bounded linear operators defined in $H$. Let $T$ be an operator in $L(H)$. The operator $T$ is called normal if it satisfies the following condition $T^*T = TT^*$, i.e., $T$ commutes with $T^*$. The class of quasi – normal operators was first introduced and studied by A brown [1] in 1953. The operator $T$ is quasi normal if $T^*T$ commutes with $T$, i.e. $T(T^*T) = (T^*T)T$ and it is denoted by $[QN]$. In 2015, Laith. K.Shaakir, and Saad. S. Marai [9], introduced the class of Quasi normal operators of order $n$ on a complex Hilbert space $H$. $T(T^*nT^n) = (T^{*n}T^n)T$. 

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A.A.S. Jibril [2], in 2008 introduced the class of n – power normal operators as a generalization of normal operators. The operator T is called n – power normal if $T^n$ commutes with $T^*$, i.e. $T^n T^* = T^* T^n$ and is denoted by [nN]. In the year 2011, Ould Ahmed Mahmoud Sid Ahmed introduced n – power quasi normal operators [3] as a generalization of quasi normal operators. The operator T is called n power quasi normal if $T^n$ commutes with $T^* T$, i.e. $T^n (T^* T) = (T^* T) T^n$ and it is denoted by [nQN]. D. Senthil Kumar [4], in 2012, introduced k – quasi normal operators as a generalization of concept of Quasi normal operators. An operator T is called k – quasi normal operator if $T(T^* T)^k = (T^* T)^k T$. k quasi – P - normal operator is defined as the generalization of k – quasi normal operator and Quasi – P – normal operator. K.M. Manikanadan and M. Athiyaman [5] in 2016, introduced k – Quasi – P normal operator. An operator T is k – quasi – P - normal operator if $(T^* T)^k (T + T^*) = (T + T^*) (T^* T)^k$ [5].

The following diagram depicts the relations among various classes of operators.

**Remark:** If $k = 1$, then n power k quasi normal operator becomes n power quasi normal operator.
The following inequalities are valid.

\[ [nN] \subset [nQN] \subset [(n,k)QN] \subset n, k QPN. \]

and

\[ [kQN] \subset [kQN] \subset [(n,k)QN] \subset n, k QPN. \]

and

\[ [(n,k)QPN] \subset [kQPN] \text{ for } n = 1 \text{ (Particular case).} \]

Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space. A transformation \(T\) is said to be measurable if \(T^{-1}(B) \in \mathcal{A}\) for \(B \in \mathcal{A}\). A measurable transformation \(T\) is said to be non-singular if \(\mu(T^{-1}(B)) = 0\) whenever \(\mu(B) = 0\) for every \(B \in \mathcal{A}\). If \(T\) is a measurable transformation then \(T^n\) is also a measurable transformation for all natural number \(n\). If \(T\) is non-singular, then we say that \(\mu(T^{-1})\) is absolutely continuous with respect to \(\mu\) and hence \(\mu(T^{-1})^n\) becomes absolutely continuous with respect to \(\mu\). Hence, by Radon-Nikodym theorem there exists a unique non-negative essentially bounded measurable function \(h_n\) such that \(\mu(T^{-1})^n(B) = \int_B h_n \, d\mu\) for \(B \in \mathcal{A}\) and \(h_n\) is called the Radon-Nikodym derivative and is denoted by \(\frac{d\mu(T^{-1})^n}{d\mu}\). Let \(\varphi\) be an essentially bounded function. The multiplication operator \(M_\varphi\) on the space \(L^2(\mu)\) induced by \(\varphi\) is given by

\[ M_\varphi f = \varphi \cdot f \quad \text{for } f \in L^2(\mu). \]

Let \(T\) be a measurable transformation on \(X\). The composition operator \(C_T\) on the space \(L^2(\mu)\) is given by

\[ C_T f = f \circ T \quad \text{for } f \in L^2(\mu). \]

Let \(C_T\) be the composition operator and \(C_T^*\) be its adjoint which is given by

\[ C_T^* f = h \cdot E(f) \circ T^{-1}. \]

**Composite multiplication operator.**

A composite multiplication operator is a linear transformation acting on a a set of complex valued \(\Sigma\)-measurable functions \(f\) of the form \(M_{u,T}(f) = C_T \, M_u \, (f) = C_T \, (uf) = (uf) \circ T = (u \circ T) \cdot (f \circ T)\), where \(u\) is a complex valued \(\Sigma\)-measurable function. In case, \(u = 1\) almost everywhere, then \(M_{u,T}\) becomes a composition operator. The adjoint of \(M_{u,T}\) is given by \(M_{u,T}^* f = uh \cdot E(f) \circ T^{-1}\).

2.0. **Characterizations on \(n\) – power - k – quasi – P – normal operator.**
Definition: 2.1
An operator $T \in L(H)$, is called $n$–power -$k$- Quasi- $P$-normal operators and it is defined as $(T^*T)^k(T^n + T^{*n}) = (T^n + T^{*n})(T^*T)^k$, where $n$ is the positive integer greater than one and $k$ is any positive integer. This operator is introduced as the generalized concept of $n$ power $k$–quasi normal operators $T^n(T^*T)^k = (T^*T)^kT^n$.

Remark: 2.2
In particular, for $n = 1$, this operator becomes $k$–quasi $P$–normal operator.

Theorem: 2.3
Every $n$ power $k$–quasi normal operator is $n$–power -$k$–quasi $P$–normal operator.

Proof:
Let $T \in L(H)$ be an $n$ power $k$ – quasi normal operator, Therefore, we have,

$T^n(T^*T)^k = (T^n + T^{*n})(T^*T)^k$ (1)

Therefore, we have,

$(T^*T)^kT^{*n} = T^{*n}(T^*T)^k$ (2)

Adding (1) and (3), we get,

$T^n(T^*T)^k + T^{*n}(T^*T)^k = (T^n + T^{*n})(T^*T)^k = (T^*T)^k(T^n + T^{*n})$

Hence $T$ is $k$ – quasi – $P$ – normal operator.

Example 2.4: The operator $T = \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}$ acting on two dimensional Hilbert space $\mathbb{C}^2$ is $2$ – power quasi $P$ normal operator but it is not $3$ – power quasi $P$ normal operator.

Solution: On substituting, $k =1$ and $n =2$ in the definition 2.1 we get,

$(T^*T)(T^2 + T^{*2}) = (T^2 + T^{*2}) (T^*T)$. On direct matrices multiplication, it can be easily verified that $(T^*T)(T^2 + T^{*2}) = \begin{pmatrix} -2 & 2i \\ -2i & -4 \end{pmatrix} = (T^2 + T^{*2}) (T^*T)$. Hence $T$ is quasi $P$ normal operator.

Further, we have, $(T^*T)(T^3 + T^{*3}) = \begin{pmatrix} i & -1 \\ -2 & -i \end{pmatrix} \neq \begin{pmatrix} -i & -2 \\ -1 & i \end{pmatrix} = (T^3 + T^{*3}) (T^*T)$. Hence $T$ is not quasi $P$ normal operator.

3.0. Characterizations on $n$ power $k$ quasi $P$ - normal composition operators

Definition:
Let the composition operator $C_S$ on $L^2(\mu)$ be unitarily equivalent to $C_T$. Then there is an unitary operator $U \in L(H)$ such that $C_S = UC_TU^*$ and $C_S^* = UC_T^*U^*$.

Theorem 3.1:
If $C_T$ is $n$ power $k$ – quasi $P$ - normal operator on a Hilbert space and $C_S$ is unitarily equivalent to $C_T$, then $C_S$ is $n$ power $k$ – quasi $P$ - normal operator.

**Proof:**

\[(C^n_S + C^n_S)(C^n_S C^n_S)^k = (UC^n_T U^* + UC^n_T U^*) (UC^n_T U^* U C_T U^*)^k \]
\[= (UC^n_T U^* + UC^n_T U^*) (UC^n_T U^*)^k \]
\[= (UC^n_T U^*) (UC^n_T U^*)^k + (UC^n_T U^*) (UC^n_T U^*)^k \]
\[= UC^n_T U^* U (C^n_T C_T)^k U^* + UC^n_T U (C^n_T C_T)^k U^* \]
\[= UC^n_T (C^n_T C_T)^k U^* + UC^n_T (C^n_T C_T)^k U^* \]
\[= U [ C^n_T + C^n_T ] (C^n_T C_T)^k U^* \]
\[= U(C^n_T C_T)^k (C^n_T + C^n_T) U^* \}

This implies that $(C^n_S + C^n_S)(C^n_S C^n_S)^k = (C^n_S C_S)^k (C^n_S + C^n_S)$.

**Proposition 3.2:**

Let $C_T \in B(L^2(\mu))$ be a composition operator. Then the following results hold.

(i) $C^n_T (C^n_T C^n_T)^k f = h. E [ h ] \circ T^{-1} . E [ h ] \circ T^{-2} ... E [ h ] \circ T^{-(n-1)}. E [ h ] \circ T^{-n} \}

(ii) $(C^n_T C^n_T)^k C^n_T f = h^k. h. E [ h ] \circ T^{-1} . E [ h ] \circ T^{-2} ... E [ h ] \circ T^{-(n-1)}. E [ h ] \circ T^{-n} \}

where $E$ is the projection of $L^2$ on to $\overline{R(C_T)}$.

**Proof:**

(i) $C^n_T (C^n_T C^n_T)^k f = C^n_T h^k f$

\[= C^n_T (C^n_T)^*(h^k. f) \]
\[= C^n_T [ h . E[h] \circ T^{-1} . E[f] \circ T^{-1}] \]
\[= C^n_T h. E [ h ] . E[f] \circ T^{-1} . E[f] \circ T^{-1} \}

(ii) $(C^n_T C^n_T)^k C^n_T f = h^k. h. E [ h ] \circ T^{-1} . E[f] \circ T^{-1}$
\[ E \circ T = T \circ E \]
\[ = C^0_T \circ E \]
\[ = C^0_T \circ E \]
\[ = C^0_T \circ E \]
\[ = C^0_T \circ E \]
\[ = C^0_T \circ E \]
\[ \cdots \]
\[ = h. E \] (the projection of \( L^2 \) on to \( R(C_T) \).)
(iii) (i) = (ii) implies that \([E[h] \circ T^{-n}]^k = h^k\).

**Theorem 3.4**

Let \(C_T \in B(L^2(\mu))\). Then \(C_T\) is in the class \([(n,k)\text{QPN}]\) if and only if \((h \circ T^n)^k, (f \circ T^n) + h.E[h] \circ T^{-1}, E[h] \circ T^{-2}, E[h] \circ T^{-(n-1)}, [E[h] \circ T^{-n}]^k, E[f] \circ T^{-n} = h^k, f \circ T^n + h^{k+1}, E[h] \circ T^{-1}, E[h] \circ T^{-2}, E[h] \circ T^{-(n-1)}, E[f] \circ T^{-n}\), where \(E\) is the projection of \(L^2\) on to \(R(C_T)\).

**Proof:**

\[C_T \in [(n,k)\text{QPN}] \Leftrightarrow (C_T^n + C_T^m)(C_T^*C_T)^k = (C_T^*C_T)^k(C_T^n + C_T^m)\]

It can be easily verified that,

\[C_T^n(C_T^*C_T)^k f = (h \circ T^n)^k, (f \circ T^n)\]

\[C_T^m(C_T^*C_T)^k f = h.E[h] \circ T^{-1}, E[h] \circ T^{-2}, E[h] \circ T^{-(n-1)}, [E[h] \circ T^{-n}]^k, E[f] \circ T^{-n}\]

\[(C_T^*C_T)^k C_T^n f = h^k, f \circ T^n\]

\[(C_T^*C_T)^k C_T^m f = h^{k+1}, E[h] \circ T^{-1}, E[h] \circ T^{-2}, E[h] \circ T^{-(n-1)}, E[f] \circ T^{-n}\]

Hence, \(C_T \in [(n,k)\text{QPN}] \Leftrightarrow C_T^n(C_T^*C_T)^k f + C_T^m(C_T^*C_T)^k f = (C_T^*C_T)^k C_T^n f + (C_T^*C_T)^k C_T^m f\)

\[\Leftrightarrow (h \circ T^n)^k, (f \circ T^n) + h.E[h] \circ T^{-1}, E[h] \circ T^{-2}, E[h] \circ T^{-(n-1)}, [E[h] \circ T^{-n}]^k, E[f] \circ T^{-n} = h^k, f \circ T^n + h^{k+1}, E[h] \circ T^{-1}, E[h] \circ T^{-2}, E[h] \circ T^{-(n-1)}, E[f] \circ T^{-n}\]

**Corollary 3.5:**

Let \(C_T \in B(L^2(\mu))\) with dense range. Then \(C_T\) is in the class \([(n,k)\text{QPN}]\) if and only if

(i) \((h \circ T^n)^k = h^k\) a.e. in the \(R(C_T^n)\).

(ii) \([E[h] \circ T^{-n}]^k = h^k\), where \(E\) is the projection of \(L^2\) on to \(R(C_T^n)\).
Proposition 3.6.
Let \( C_T \in B(L^2(\mu)) \) be a composition operator. Then the following results hold.

(i) \( C_T^n (C_T C_T^*)^k f = (h \circ T^{n+1})^k \cdot E(f) \circ T^n \)

(ii) \( (C_T C_T^*)^k C_T^n f = (h \circ T)^k \cdot (f \circ T^n) \).

Proof:

(i) \( C_T^n (C_T C_T^*)^k f = C_T^n [ (h \circ T)^k \cdot E(f) ] \)

\[ = [(h \circ T)^k \cdot E(f)] \circ T^n \]

\[ = (h \circ T^{n+1})^k \cdot E(f) \circ T^n \]

(ii) \( (C_T C_T^*)^k C_T^n f = (C_T C_T^*)^k [ f \circ T^n ] \)

\[ = (C_T C_T^*)^{k-1} C_T C_T^* [ f \circ T^n ] \]

\[ = (C_T C_T^*)^{k-1} C_T h \cdot E [ f \circ T^n ] \circ T^{-1} \]

\[ = (C_T C_T^*)^{k-1} C_T [ h \cdot f \circ T^{n-1} ] \]

\[ = (C_T C_T^*)^{k-1} [ h \cdot f \circ T^{n-1} ] \circ T \]

\[ = (C_T C_T^*)^{k-1} [ h \circ T \cdot f \circ T^n ] \]

\[ = (C_T C_T^*)^{k-2} C_T h \cdot E [ h \cdot f \circ T^n ] \circ T^{-1} \]

\[ = (C_T C_T^*)^{k-2} C_T h \cdot f \circ T^{n-1} \]

\[ = (C_T C_T^*)^{k-2} [ h \cdot h \cdot f \circ T^{n-1} ] \circ T \]

\[ = (C_T C_T^*)^{k-2} h \circ T \cdot h \circ T \cdot f \circ T^n \]

\[ = (C_T C_T^*)^{k-2} [(h \circ T)^2 \cdot f \circ T^n] \]

\[ \cdots \]

\[ \cdots \]

\[ = (h \circ T)^k \cdot f \circ T^n \]
Theorem 3.7

Let \( C_T^* \in L^2(\mu) \). Then \( C_T^* \) is in the class \([(n, k)QPN]\) if and only if

\[
h \circ E[h] \circ T^{-1} \circ E[h] \circ T^{-2} \circ ... \circ E[h] \circ T^{-(n-2)} \circ E[h] \circ T^{-(n-1)} \circ E[f] \circ T^{-n} + (h \circ T^{n+1}) \circ E(f) \circ T^n = (h \circ T)^k \circ E[h] \circ E[h] \circ E[h] \circ E[h] \circ ... \circ E[h] \circ E[f] \circ T^{-n} \circ f + (h \circ T)^k \circ E[f] \circ T^n.
\]

**Proof:**

\( C_T^* \in [(n, k)QPN] \iff (C_T^n + C_T^k)(C_T^* C_T^*)^k = (C_T^n C_T^*)^k (C_T^n + C_T^k) \)

It can be easily verified that,

\[
(C_T C_T^*)^k C_T^n f = (h \circ T)^k \circ E[h] \circ T^{-1} \circ E[h] \circ T^{-2} \circ ... \circ E[h] \circ T^{-(n-1)} \circ E[f] \circ T^{-n}
\]

\((C_T C_T^*)^k C_T^n f = (h \circ T)^k \circ E[h] \circ T^{-1} \circ E[h] \circ T^{-2} \circ ... \circ E[h] \circ T^{-(n-1)} \circ E[f] \circ T^{-n}
\]

From proposition 3.5, it is well known that, \( C_T^k (C_T C_T^*)^k f = (h \circ T^{n+1}) \circ E(f) \circ T^n \) and

\[
(C_T C_T^*)^k C_T^k f = (h \circ T)^k \circ f \circ T^n.
\]

Hence,

\[
C_T^* \in [(n, k)QPN] \iff C_T^n (C_T C_T^*)^k f + C_T^k (C_T^* C_T)^k f = (C_T C_T^*)^k C_T^n f + (C_T C_T^*)^k C_T^k f
\]

\[
\iff h \circ E[h] \circ T^{-1} \circ E[h] \circ T^{-2} \circ ... \circ E[h] \circ T^{-(n-2)} \circ E[h] \circ T^{-(n-1)} \circ E[f] \circ T^{-n} + (h \circ T^{n+1}) \circ E(f) \circ T^n = (h \circ T)^k \circ E[h] \circ T^{-1} \circ E[h] \circ T^{-2} ... \circ E[h] \circ T^{-n} \circ f + (h \circ T)^k \circ E[f] \circ T^n.
\]

**4.0 Characterizations on n power k – Quasi normal operators**

**Definition 4.1:** An operator \( T \in L(H) \) is said to be n power k quasi normal operator if

\[
T^n (T^* T)^k = (T^* T)^k T^n \text{for } n \in N \text{ and it is denoted by } [(n, k)QN].
\]
Theorem: 4.2
Every k–quasi normal operator is k–quasi–P–normal operator.

Proof:
Let \( T \in L(H) \) be a k–quasi normal operator.

Therefore, we have,
\[
(T^*T)^kT = (T^*T)^k \quad \ldots \quad (1)
\]
\[
(T^*T)^kT^* = T^* (T^*T)^k \quad \ldots \quad (2)
\]
\[
T^* (T^*T)^k = (T^*T)^kT^* \quad \ldots \quad (3)
\]

Adding (1) and (3), we get,
\[
T(T^*T)^k + T^* (T^*T)^k = (T^*T)^k T + (T^*T)^k T^*
\]
\[
(T + T^*) (T^*T)^k = (T^*T)^k (T + T^*)
\]

Hence \( T \) is k–quasi–P–normal operator.

Theorem: 4.3
Every quasi normal operator is quasi–P–normal operator.

Proof:
Let \( T \in L(H) \) be a k–quasi normal operator.

\[
T(T^*T) = (T^*T)T \quad \ldots \quad (1)
\]
\[
(T^*T)^k = T^* (T^*T) \quad \ldots \quad (2)
\]
\[
T^* (T^*T)^k = (T^*T)^kT^* \quad \ldots \quad (3)
\]

Adding (1) and (3), we get,
\[
(T + T^*) (T^*T) = (T^*T) (T + T^*)
\]

Hence \( T \) is quasi–P–normal operator.

Definition 4.4 Subnormal operator

An operator \( T \) on a Hilbert space \( H \) is sub normal if there exists a Hilbert space \( K \) containing \( H \) and a normal operator on \( K \) such that \( H \) is \( N \)–invariant. i.e. \( N(H) \subseteq H \) and \( T \) is restriction of \( N \) to \( H \). i.e. \( T = N|H \).

In other words an operator \( T \in B(H) \) is sub normal if \( H \) is a subspace of a larger Hilbert space \( K \) so that \( K = H \oplus H^\perp \) and \( N = \begin{pmatrix} T & X \\ O & Y \end{pmatrix} : H \oplus H^\perp \rightarrow H \oplus H^\perp \) is a normal operator in \( B(K) \) for some \( X \in B(H^\perp, H) \) and some \( Y \in B(H^\perp). \)
**Proposition 4.5:** Every \( n \) power \( k \) quasi normal operator is subnormal operator.

**Proof**

Since \( T \) is \( n \) power \( k \) quasi normal operator, we have \((T^*T)^k\) commutes with \( T^n \) and \( T^n = T^n \). So \( N[(T^*T)^k] \) reduces \( T^n \). But \( N[(T^*T)^k] = N(T^n) \). Therefore \( N(T^n) \) reduces \( (T^n) \). Hence \( T \) is subnormal operator.

**Corollary 4.6.** Every \( k \) quasi normal operator is subnormal operator.

The result is obvious by taking \( n = 1 \) in the proposition 1.8

### 5.0 \( n \) power \( k \) quasi normal composition operators

**Theorem 5.1:**

If \( C_T \) is \( n \) power \( k \) – quasi normal operator on a Hilbert space and \( C_S \) is unitarily equivalent to \( C_T \), then \( C_S \) is \( n \) power \( k \) – quasi normal operator.

**Proof:**

\[
(C_S^n C_S^k C_T^n C_T^k)^k = (UC_T^n U^*) (UC_T^n U^* UC_T^n U^*)^k \\
= UC_T^n U^* (UC_T^n C_T^n U^*)^k \\
= UC_T^n U^* UC_T^n (C_T^n C_T^n)^k U^* \\
= UC_T^n (C_T^n C_T^n)^k U^* \\
= UC_T^n (C_T^n C_T^n)^k C_T^n U^* \\
\]

This implies that \( C_S^n (C_S^k C_S^n)^k = (C_S^k C_S^n)^k C_S^n \).

The following lemmas of [11, 12, 13] play an important role in the following theorems.

**Lemma 5.2:** [11, 12, 13]. Let \( P \) be the projection operator of \( L^2(\lambda) \) onto \( \overline{R(C_T)} \).

(i) \( C_T^* C_T f = h f \) and \( C_T C_T^* f = h \circ T P f \) for every \( f \in L^2(\lambda) \).
(ii) \( \overline{R(C_T^\mu)} = \{ f \in L^2(\lambda) : f \text{ is } T^{-1}(\Sigma) \text{ measurable} \} \).

(iii) If \( f \) is \( T^{-1}(\Sigma) \text{ measurable} \) and \( g \) and \( fg \) belong to \( L^2(\lambda) \), then \( P(fg) = f P(g) \) (\( f \) need not be in \( L^2(\lambda) \)).

(iv) \( (C_T C_T^*)^k f = h^k f \)

(v) \( (C_T C_T^*)^k f = (h \circ T)^k P f \)

(vi) \( E \) is the identity operator on \( L^2(\lambda) \) if and only if \( T^{-1}(\Sigma) = \Sigma \).

Gupta A. and Bhatia N [10] proved the following theorem.

**Theorem 5.3** [10]

Let \( C_T \in B(L^2(\mu)) \) be a composition operator. Then \( C_T \) is n quasi normal if and only if \( h \circ T^n = h \) a.e. for every \( n \in \mathbb{N} \).

We prove that this theorem is valid only for every \( f \in L^2(\mu) \) has dense range in \( \overline{R(C_T^\mu)} \)

the restate the above theorem as follows.

Let \( C_T \in B(L^2(\mu)) \) be a composition operator. Then \( C_T \) is n quasi normal if and only if

(i) \( h \circ T^n \). \( f \circ T^n = h \cdot f \circ T^n \) a.e. in \( R(C_T^n) \) and for every \( n \in \mathbb{N} \).

(ii) Let \( C_T \in B(L^2(\mu)) \) with dense range. Then \( h \circ T^n = h \) in \( \overline{R(C_T^n)} \) and for every \( n \in \mathbb{N} \).

**Proof:** (i) For \( f \in L^2(\mu) \).

\[
C_T^n(C_T C_T^*) f = C_T^n (hf) = (h \cdot f) \circ T^n = h \circ T^n \cdot f \circ T^n.
\]

\[
(C_T C_T^*)^k f = (C_T C_T^*)(f \circ T^n) = h \cdot f \circ T^n
\]

Suppose \( C_T \) is n quasi normal. Then \( C_T^n(C_T C_T^*) f = (C_T C_T^*)^k f \).

This implies that \( h \circ T^n \cdot f \circ T^n = h \cdot f \circ T^n \) a.e. in \( R(C_T^n) \)

(ii) Suppose \( C_T \in B(L^2(\mu)) \) with dense range, then, we have \( h \circ T^n = h \).

**Theorem 5.4**

Let \( C_T \in B(L^2(\mu)) \). Then \( C_T \) is in the class \( [(n,k)QN] \) if and only if \( (h \circ T^n)^k \cdot (f \circ T^n) = h^k \cdot f \circ T^n \) in \( R(C_T^n) \).

**Proof:**

\[
C_T \in [(n,k)QN] \iff C_T^n (C_T C_T^*)^k = (C_T C_T^*)^k C_T^n
\]

Consider \( C_T^n (C_T C_T^*)^k f = C_T^n (C_T C_T^*)^{k-1} C_T C_T^* f \)
\[ = C^n_T \left( C_T \ast C_T \right)^{k-1} C_T \ast (f \circ T) \]
\[ = C^n_T \left( C_T \ast C_T \right)^{k-1} h. E[f \circ T] \circ T^{-1} \]
\[ = C^n_T \left( C_T \ast C_T \right)^{k-1} h. f \]
\[ = C^n_T \left( C_T \ast C_T \right)^{k-2} h. h. f \]
\[ = \ldots \ldots \]
\[ = C^n_T h^k f \]
\[ = C^{n-1}_T (h^k. f) \]
\[ = C^{n-1}_T (h^k. f) \circ T \]
\[ = C^{n-1}_T (h \circ T. h \circ T \ldots k \text{ times.}) \circ T \]
\[ = C^{n-1}_T (h \circ T. h \circ T \ldots k \text{ times.}) \circ T \]
\[ = (h \circ T)^{k}\left( f \circ T \right) \]
\[ = C^{n-2}_T \left( (h \circ T)^{k}. (f \circ T) \right) \circ T \]
\[ = C^{n-2}_T \left( (h \circ T)^{k}. (f \circ T) \right) \circ T \]
\[ = C^{n-2}_T \left( (h \circ T)^{k} \ldots k \text{ times.} (f \circ T) \right) \circ T \]
\[ = C^{n-2}_T \left( (h \circ T^2)^{k} \ldots k \text{ times.} (f \circ T^2) \right) \]
\[ = \ldots \ldots \]
\[ = \ldots \ldots \]
\[ = (h \circ T^n)^{k}. (f \circ T^n) \]

Consider \( (C_T \ast C_T)^k C^n_T f \)

\[ = (C_T \ast C_T)^k (f \circ T^n) \]
\[ = (C_T \ast C_T)^{k-1} C_T \ast (f \circ T^n) \]
\[ = (C_T \ast C_T)^{k-1} C_T \ast (f \circ T^{n+1}) \]
\[ = (C_T \ast C_T)^{k-1} h. E[f \circ T^{n+1}] \circ T^{-1} \]
\[ = (C_T \ast C_T)^{k-1} h. f \circ T^n \]
\[ = (C_T \ast C_T)^{k-2} h. f \circ T^n \]
\[ = \ldots \ldots \]
\[ = h^k. f \circ T^n \]
\[ C_T \in [(n, k)QN] \iff C_T^n (C_T \ast C_T)^k = (C_T \ast C_T)^k C_T^n \]

\[ \iff (h \circ T^n)^k \cdot (f \circ T^n) = h^k \cdot f \circ T^n \]

**Corollary: 5.5**

Let \( C_T \in L^2(\mu) \) with dense range. Then \( C_T \) is in the class \([n, k)QN]\ if and only if \( (h \circ T^n)^k = h^k \) in \( \overline{R(C_T^n)} \).

**Remark 5.6 [11, Theorem 2.2]**

\[ C_T^n f = h \cdot E[h] \circ T^{-1} \cdot E[h] \circ T^{-2} \ldots E[h] \circ T^{-(n-1)} E(f) \circ T^{-n} \]

**Theorem 5.7**

Let \( C_T \ast \in B(L^2(\mu)) \). Then \( C_T \ast \) is in the class \([n, k)QN]\ if and only if \( h \cdot E[h] \circ T^{-1} \cdot E[h] \circ T^{-2} \ldots E[h] \circ T^{-(n-2)} E[h] \circ T^{-(n-1)} [E[h] \circ T^{-(n-1)}]^k E(f) \circ T^{-n} \) holds in the \( \overline{R(C_T^n)} \).

**Proof:**

\[ C_T \ast \in [(n, k)QN] \iff C_T^n (C_T \ast C_T \ast)^k = (C_T \ast C_T \ast)^k C_T^n \]

Consider \( C_T^n (C_T \ast C_T \ast)^k f = C_T^n \cdot [h \circ T^k] \cdot E(f) \]

\[ = C_T^{n-1} \cdot [h \circ T^k] \cdot E(f) \]

\[ = C_T^{n-1} h \cdot E[h] \circ T^{-(n-2)} E(f) \circ T^{-(n-1)} \]

\[ = C_T^{n-1} h \cdot h^k \cdot E(f) \circ T^{-1} \]

\[ = C_T^{n-1} h^k \cdot E(f) \circ T^{-1} \]

\[ = C_T^{n-2} h \cdot E[h] \circ T^{-1} E(f) \circ T^{-1} \]

\[ = C_T^{n-2} h \cdot E[h] \circ T^{-1} E(f) \circ T^{-2} \]

\[ = C_T^{n-3} h \cdot E[h] \circ T^{-1} E(f) \circ T^{-3} \]

\[ = C_T^{n-4} h \cdot E[h] \circ T^{-4} \]

\[ \ldots \]
Consider \((C_T C_T^*)^kC_T^n f\) = \((C_T C_T^*)^kC_T^n h\). Hence, \(C_T^* \in [(n,k)QN] \iff C_T^n (C_T C_T^*)^k f = (C_T C_T^*)^k C_T^n f\).

\[\begin{align*}
\text{Corollary 5.9} & \\
\text{Let } C_T^* & \in B(L^2(\mu)) \text{ with dense range. Then } C_T^* \text{ is in the class } [(n,k)QN] \text{ if and only if } h_n (h \circ T)^k \circ T^{-n} = (h \circ T)^k \circ T^{-n}. \text{ for every } f \in L^2(\mu) \text{ in the } R(C_T^*) \\
\text{The proof is obvious when } C_T^n f = h \circ T^{-1}. h \circ T^{-2} \cdots \circ T^{-n} \text{ for every } f \in L^2(\mu) \text{ in the } R(C_T^*) \\
\text{Corollary 5.9} & 
\end{align*}\]
An operator $C_T$ is $n$-power $k$-quasi normal if and only if $C_T^*$ is $n$-power $k$-quasi normal operator.

Proof: 
Let $C_T$ be a $n$-power $k$-quasi normal composition operator. Then 

$$C_T \in [(n,k)QN] \iff C_T^n (C_T^* C_T)^k = (C_T^* C_T)^k C_T^n$$

Taking adjoint on both sides

$$C_T^* C_T^n = C_T^n C_T^* C_T^k$$

$C_T^*$ is $n$-power $k$-quasi normal operator.

### 6.0 $n$ power $k$ quasi normal weighted composition operators

The following proposition of [10] plays an important role in the following theorems.

**Proposition: 6.1 [10, Proposition 10.3]**

For $\varphi \geq 0$, (i) $W^* W f = h. E[\varphi^2] \circ T^{-1} f$ (ii) $W W^* f = \varphi(h \circ T)E(\varphi f)$

**Theorem 6.2:** Let $W$ be weighted composition operator. Then for $\varphi \geq 0$ the following results hold.

(i) $(W^* W)^k f = h^k. [E[\varphi^2] \circ T^{-1}]^k f$

(ii) $(W W^*)^k f = \varphi(h \circ T)^k. [E[\varphi^2]]^{k-1}. E[\varphi. f]$

**Theorem 6.3:**

Let $W$ be a weighted composition operator on $L^2(\mu)$. Then $W \in [n(k)QN]$ if and only if 

$$\varphi_n(h \circ T^n)^k. [E[\varphi^2] \circ T^{-1}]^k. f \circ T^n = h^k. [E[\varphi^2] \circ T^{-1}]^{k-1}. E[\varphi. \varphi_{n+1}] \circ T^{-1}. f \circ T^n$$

in the $R(CT^n)$. 

**Proof:** 
$W \in [(n,k)kQN] \iff W^n (W^* W)^k = (W^* W)^k W^n$

Consider, $W^n (W^* W)^k f = W^n [h^k. [E[\varphi^2] \circ T^{-1}]^k f]$ 

$= W^{n-1} W [h^k. [E[\varphi^2] \circ T^{-1}]^k f]$

$= W^{n-1} \varphi. [h^k. [E[\varphi^2] \circ T^{-1}]^k f] \circ T$

$= W^{n-1} \varphi. (h \circ T)^k. E[\varphi^2]^k. f \circ T$

$= W^{n-2} W [\varphi. (h \circ T)^k. E[\varphi^2]^k. f \circ T]$

$= W^{n-2} \varphi. [\varphi. (h \circ T)^k. E[\varphi^2]^k. f \circ T] \circ T$
\[ = W^{n-2} \varphi \circ T. (h \circ T^2)^k. [E[\varphi^2] \circ T] \circ T \]
\[ = W^{n-3} \varphi \circ T. \varphi \circ T^2. (h \circ T^3)^k. [E[\varphi^2] \circ T^2] \circ T \circ T^3 \]
\[
......
\]
\[
......
\]
\[ = \varphi \circ T. \varphi \circ T^2 ... \varphi \circ T^{n-1} (h \circ T^n)^k. [E[\varphi^2] \circ T^{n-1}] \circ T \circ T^n \]
\[= \varphi_n (h \circ T^n)^k. [E[\varphi^2] \circ T^{n-1}] \circ T \circ T^n \]

Consider, \((W^W)^k W^n f = (W^W)^k \varphi_n \circ T^n\)
\[ = (W^W)^k W^W (\varphi_n \circ T^n) \]
\[ = (W^W)^k W^W (\varphi_n \circ T^n) \]
\[ = (W^W)^k-1 W^W (\varphi_n \circ T^n+1) \]
\[ = (W^W)^k-1 h. E \{ \varphi. \varphi_n+1 \circ T^n+1 \} \circ T^{-1} \]
\[ = (W^W)^k-1 h. E \{ \varphi. \varphi_n+1 \} \circ T^{-1, f \circ T^n} \]
\[ = (W^W)^k-2 \varphi \circ T^{-1, f \circ T^n} \circ T \]
\[ = (W^W)^k-2 \varphi \circ T^{-1, f \circ T^n} \circ T \]
\[ = (W^W)^k-2 \varphi \circ T^{-1, f \circ T^n} \circ T \]
\[ = (W^W)^k-2 \varphi \circ T^{-1, f \circ T^n} \circ T \]
\[ = (W^W)^k-3 h. E[\varphi^2] \circ T^{-1}. h. E[\varphi^2] \circ T^{-1}. h. E[\varphi. \varphi_n+1] \circ T^{-1, f \circ T^n} \]
\[ = h^k. [E[\varphi^2] \circ T^{-1}]^{k-1}. E[\varphi. \varphi_n+1] \circ T^{-1}. f \circ T^n \]

Therefore, \(W \in [n, k) QN \) \iff \( W^n (W^W)^k f = (W^W)^k W^n f \)
\[ \iff \varphi_n (h \circ T^n)^k. [E[\varphi^2] \circ T^{n-1}]^k \circ T^n = h^k. [E[\varphi^2] \circ T^{-1}]^{k-1}. E[\varphi. \varphi_n+1] \circ T^{-1}. f \circ T^n \]

**Corollary 6.4**

Let \( W \) be a weighted composition operator on \( L^2(\mu) \) with dense range. Then \( W \in [n k QN] \) if and only if
\[ \varphi_n (h \circ T^n)^k. [E[\varphi^2] \circ T^{n-1}]^k = h^k. [E[\varphi^2] \circ T^{-1}]^{k-1}. E[\varphi. \varphi_n+1] \circ T^{-1}. \]
Theorem 6.5:

Let $W$ be a weighted composition operator on $L^2(\mu)$. Then $W^* \in [nkQN]$ if and only if

$$W^* = h. E[\varphi] \circ T^{-1}. E[h] \circ T^{-1}. E[\varphi] \circ T^{-2}. E[h] \circ T^{-2} \circ E[\varphi] \circ T^{-3}. E[h] \circ T^{-3} \circ \ldots \circ E[\varphi] \circ T^{-(n-1)}. E[h] \circ T^{-(n-1)}E[\varphi^2] \circ T^{-n}. E[h] \circ T^{-(n-1)} \circ E[\varphi^2] \circ T^{-n} \circ \ldots \circ E[\varphi] \circ T^{-n}.$$

**Proof:**

$$W^* \in [nkQN] \iff W^{*n}(WW^*)^k = (WW^*)^k \ W^{*n}$$

$$W^{*n}(WW^*)^k f = W^{*n} [ \varphi \circ (h \circ T)^k. [E[\varphi^2]]^{k-1}. E[\varphi, f] ]$$

$$= W^{*n-1} h. E[\varphi \circ (h \circ T)^k. [E[\varphi^2]]^{k-1}. E[\varphi, f] ] \circ T^{-1}$$

$$= W^{*n-1} h. E[\varphi^2] \circ T^{-1}. h^k. [E[\varphi^2] \circ T^{-1}]^{k-1}. E[\varphi, f] \circ T^{-1}$$

$$= W^{*n-2} h. E[\varphi \circ T^{-1}. E[h] \circ T^{-1}. E[\varphi^2] \circ T^{-2}. [E[h] \circ T^{-1}]^k. [E[\varphi^2] \circ T^{-2}]^{k-1}. E[\varphi, f] \circ T^{-1}$$

$$= W^{*n-3} h. E[\varphi] \circ T^{-1}. E[h] \circ T^{-1}. E[\varphi] \circ T^{-2}. E[h] \circ T^{-2}. E[\varphi^2] \circ T^{-3}. [E[\varphi^2] \circ T^{-3}]^{k-1}. E[\varphi, f] \circ T^{-3}$$

$$= W^{*n-4} h. E[\varphi] \circ T^{-1}. E[h] \circ T^{-1}. E[\varphi] \circ T^{-2}. E[h] \circ T^{-2}. E[\varphi^2] \circ T^{-3}. [E[\varphi^2] \circ T^{-3}]^{k-1}. E[\varphi, f] \circ T^{-4}$$

$$= \ldots$$

$$= h. E[\varphi] \circ T^{-1}. E[h] \circ T^{-1}. E[\varphi] \circ T^{-2}. E[h] \circ T^{-2} \circ E[\varphi] \circ T^{-3} \circ E[h] \circ T^{-3} \circ \ldots \circ E[\varphi] \circ T^{-(n-1)}. E[h] \circ T^{-(n-1)}E[\varphi^2] \circ T^{-n}. E[h] \circ T^{-(n-1)} \circ E[\varphi^2] \circ T^{-n} \circ \ldots \circ E[\varphi] \circ T^{-n}.$$

7.0. **Characterization of class $[n,k)QN$ composite multiplication operators.**

The following theorem gives the characterization of $n - \text{power k - Quasi normal composite multiplication operators.}$
Theorem 7.1: Let $M_u, T$ on $L^2(\lambda)$ be a composite multiplication operator, then for $\lambda \geq 0$, Then the following results hold.

(i) $M^nf = (u \circ T). (u \circ T^2). (u \circ T^3) \ldots (u \circ T^n). (f \circ T^n)$. [11, Theorem 5.14]

(ii) $(M^*_u M^*_T)^k f = u^2. h^k. \left[ E[(u \circ T)(u \circ T)] \circ T^{-1} \right]^{k-1}. f$

(iii) $(M^*_u M^*_T)^k f = (u \circ T)^2. (h \circ T)^k. \left[ E[(u \circ T)(u \circ T)] \right]^{k-1}. E(f)$

Proof:

(ii) Consider, $(M^*_u M^*_T)^k f = (M^*_u M^*_T)^k f$

$= (M^*_u M^*_T)^k \cdot M^*_u M^*_T f$

$= (M^*_u M^*_T)^{k-1} M^*_u M^*_T (u \circ T). (f \circ T)$

$= (M^*_u M^*_T)^{k-1} u \cdot h \cdot E[(u \circ T). (f \circ T)] \circ T^{-1}$

$= (M^*_u M^*_T)^{k-1} u \cdot h \cdot u \cdot f$

$= (M^*_u M^*_T)^{k-1} u^2 \cdot h \cdot f$

$= (M^*_u M^*_T)^{k-2} u^2 \cdot h^2 \cdot E[(u \circ T)(u \circ T)] \circ T^{-1} \cdot f$

$= (M^*_u M^*_T)^{k-3} u^2 \cdot h^3 \cdot E[(u \circ T)(u \circ T)] \circ T^{-1} \cdot f$

$= \ldots \ldots \ldots$

$= u^2. h^k. \left[ E[(u \circ T)(u \circ T)] \circ T^{-1} \right]^{k-1}. f$

(iii) $(M^*_u M^*_T)^k f = (M^*_u M^*_T)^k \cdot M^*_u M^*_T f$

$= (M^*_u M^*_T)^{k-1} M^*_u M^*_T u \cdot h \cdot E(f) \circ T^{-1}$

$= (M^*_u M^*_T)^{k-1} (u \circ T). [u \cdot h \cdot E(f) \circ T^{-1}] \circ T$
\[ (M_{u,T} M_{u,T}^*)^{k-1} (u \circ T). (u \circ T). (h \circ T). E(f) \]
\[ = (M_{u,T} M_{u,T}^*)^{k-1} (u \circ T)^2. (h \circ T). E(f) \]
\[ = (M_{u,T} M_{u,T}^*)^{k-2} (u \circ T)^2. (h \circ T)^2. E[(u \circ T). (u \circ T)]. E(f) \]
\[ = (M_{u,T} M_{u,T}^*)^{k-3} [ (u \circ T)^2. (h \circ T)^3. E[(u \circ T). (u \circ T)]^2. E(f) ] \]

\[ \ldots \ldots \]

\[ = (u \circ T)^2. (h \circ T)^k. [E[(u \circ T). (u \circ T)]^{k-1}. E(f) \]

**Theorem 7.2:** Let \( M_{u,T} \) be a composite multiplication operator on \( L^2(\lambda) \). Then for \( \lambda \geq 0 \) \( M_{u,T} \) is an \( n \)-power \( k \) quasi normal operator if and only if \( (u \circ T), (u \circ T^2), (u \circ T^3) \ldots (u \circ T^{n-1}), (u \circ T^n)^3, (h \circ T^n)^k, E[(u \circ T)(u \circ T)] \circ T^{n-1}. f \circ T^n = u, h. [E[(u \circ T)(u \circ T)] \circ T^{n-1}. f \circ T^n \text{ in the range of } C^n_T. \]

**Proof:** \( M_{u,T} \) is an \( n \)-power \( k \) Quasi normal operator.

\[ \iff \quad M_{u,T}^n (M_{u,T}^* M_{u,T})^k = (M_{u,T}^* M_{u,T})^k M_{u,T}^n f \]

Consider, \( M_{u,T}^n (M_{u,T}^* M_{u,T})^k f = M_{u,T}^n u^2. h^k. [E[(u \circ T)(u \circ T)] \circ T^{-1} ]^{k-1}. f. \]

\[ = M_{u,T}^{n-1} M_{u,T} [u^2. h^k. [E[(u \circ T)(u \circ T)] \circ T^{-1} ]^{k-1}. f] \]
\[ = M_{u,T}^{n-1} (u \circ T). [u^2. h^k. [E[(u \circ T)(u \circ T)] \circ T^{-1} ]^{k-1}. f] \circ T \]
\[ = M_{u,T}^{n-1} (u \circ T)^2. (h \circ T)^k. [E[(u \circ T)(u \circ T)] \circ T^{-1} ]^{k-1}. f \circ T \]
\[ = M_{u,T}^{n-1} (u \circ T)^3. (h \circ T)^k. [E[(u \circ T)(u \circ T)] \circ T^{-1} ]^{k-1}. f \circ T \]
\[ = M_{u,T}^{n-2} M_{u,T} [(u \circ T)^3. (h \circ T)^k. [E[(u \circ T)(u \circ T)] \circ T^{-1} ]^{k-1}. f \circ T ] \]
\[ = M_{u,T}^{n-2} (u \circ T). [(u \circ T)^3. (h \circ T)^k. [E[(u \circ T)(u \circ T)] \circ T^{-1} ]^{k-1}. f \circ T ] \]
\[
= M_{u,T}^{n-2} \circ (u \circ T). (u \circ T^2)^3. (h \circ T^2)^k. \left[ E \left[ (u \circ T)(u \circ T) \right] \circ T \right]^{k-1}. f \circ T^2
\]
\[
= M_{u,T}^{n-3} \circ (u \circ T). (u \circ T^2). (u \circ T^3)^3. (h \circ T^3)^k. \left[ E \left[ (u \circ T)(u \circ T) \right] \circ T^2 \right]^{k-1}. f \circ T^3
\]
\[
= M_{u,T}^{n-4} \circ (u \circ T). (u \circ T^2). (u \circ T^3). (h \circ T^4)^3. (h \circ T^4)^k. \left[ E \left[ (u \circ T)(u \circ T) \right] \circ T^3 \right]^{k-1}. f \circ T^4
\]

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\[
= (u \circ T). (u \circ T^2). (u \circ T^3) \ldots (u \circ T^{n-1}) \circ (u \circ T^n)^3. (h \circ T^n)^k. \left[ E \left[ (u \circ T)(u \circ T) \right] \circ T^{n-1} \right]^{k-1}. f \circ T^n
\]

Consider, \((M_{u,T}^* M_{u,T})^k f = (M_{u,T}^* M_{u,T})^k (u \circ T), (u \circ T^2), (u \circ T^3) \ldots (u \circ T^n) (f \circ T^n)\)

\[
= (M_{u,T}^* M_{u,T})^{k-1} M_{u,T}^* \left[ (u \circ T). (u \circ T^2). (u \circ T^3) \ldots (u \circ T^n) (f \circ T^n) \right]
\]
\[
= (M_{u,T}^* M_{u,T})^{k-1} M_{u,T}^* \left[ \left( (u \circ T). (u \circ T^2). (u \circ T^3) \ldots (u \circ T^n) (f \circ T^n) \right) \circ T \right]
\]
\[
= (M_{u,T}^* M_{u,T})^{k-1} M_{u,T}^* (u \circ T). (u \circ T^2). (u \circ T^3). (u \circ T^4) \ldots (u \circ T^{n+1}) (f \circ T^{n+1})
\]
\[
= (M_{u,T}^* M_{u,T})^{k-2} u. h. E[(u \circ T). (u \circ T^2). (u \circ T^3) \ldots (u \circ T^{n+1}) (f \circ T^{n+1})] \circ T^{-1}
\]
\[
= (M_{u,T}^* M_{u,T})^{k-2} u. h. E[(u \circ T). (u \circ T^2). (u \circ T^3) \ldots (u \circ T^{n+1})] \circ T^{-1}. f \circ T^n
\]
\[
= (M_{u,T}^* M_{u,T})^{k-3} u. h. E[(u \circ T). (u \circ T^2). (u \circ T^3) \ldots (u \circ T^{n+1})] \circ T^{-1}. f \circ T^n
\]

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\[
= u. h. \left[ E \left[ (u \circ T). (u \circ T^2) \right] \circ T^{-1} \right]^{k-1}. h^{k-1}. E \left[ (u \circ T). (u \circ T^2). (u \circ T^3) \ldots (u \circ T^{n+1}) \circ T^{-1}. f \circ T^n \right]
\]

\[
M_{u,T} \in [(n, k) QN] \Leftrightarrow M_{u,T}^{n-k} (M_{u,T}^* M_{u,T})^k = (M_{u,T}^* M_{u,T})^k M_{u,T}^n f
\]
\[ \Leftrightarrow (u \circ T) \cdot (u \circ T^2) \cdot (u \circ T^3) \cdots (u \circ T^n) \cdot (h \circ T^n)^k \cdot \left[ E \left[ (u \circ T)(u \circ T) \right] \circ T^{n-1} \right] \cdot f \circ T^n = u \cdot h \cdot E[ (u \circ T)(u \circ T) \circ T^{n-1}]^k \cdot h^{k-1} \cdot h \cdot E[ (u \circ T)(u \circ T^2) \cdot (u \circ T^3) \cdot (u \circ T^4) \cdots (u \circ T^{n+1})] \circ T^{-1} \cdot f \circ T^n. \]

**Corollary 7.3:** Let \( M_{u,T} \) be a composite multiplication operator on \( L^2(\lambda) \) with dense range. Then for \( \lambda \geq 0 \) \( M_{u,T} \) is an \( n \)-power \( k \) quasi normal operator if and only if \( (u \circ T)(u \circ T^2)(u \circ T^3) \cdots (u \circ T^{n-1})(u \circ T^n)^k \cdot \left[ E[ (u \circ T)(u \circ T)] \circ T^{n-1} \right]_{k=1}^k = u \cdot h \cdot E[ (u \circ T)(u \circ T^2)(u \circ T^3)(u \circ T^4) \cdots (u \circ T^{n+1})] \circ T^{-1}. \)

The proof is immediate consequence of previous theorem and dense range.

**Remark:** [11, Theorem 5.15]

\[ M_{u,T}^n f = u \cdot h \cdot E[u \cdot T^{-1}] \cdots E[u \cdot T^{-n}] \cdot E(f) \circ T^{-n}. \]

**Definition 7.3** \( M_{u,T}^n \) on \( L^2(\lambda) \) is \( n \)-power \( k \) quasi normal normal operator if

\[ M_{u,T}^n \cdot (M_{u,T}^n \cdot M_{u,T})^k = (M_{u,T}^n \cdot M_{u,T})^k \cdot M_{u,T}^n. \]

**Theorem 7.4:** Let \( M_{u,T} \) on \( L^2(\lambda) \) be a composite multiplication operator. Then for \( \lambda > 0 \) \( M_{u,T}^* \) is an \( n \)-power \( k \) quasi normal operator if and only if

\[ u \cdot h \cdot E[u \cdot h] \circ T^{-1} \cdot E[u \cdot h] \circ T^{-2} \cdots E[u \cdot h] \circ T^{-(n-1)} \cdot E(f) \circ T^{-n}. \]

**Proof:** The proof is obtained by using the definition 7.3, the result (iii) in theorem 7.1, and the remark: [11, Theorem 5.15].

**8.0 Characterizations on weighted shift operator.**

A weighted shift \( T[8] \) with decreasing weighted sequence \( (\alpha_n) \) is defined by \( T(e_n) = \alpha_ne_{n+1} \) and its adjoint \( T^* \) is also weighted shift and it is defined by \( T^*(e_n) = \alpha_{n-1}e_{n-1} \).

**Theorem 8.1** A weighted shift \( T \) with decreasing weighted sequence \( (\alpha_n) \) is \( n \)-power \( k \) quasi normal operator if and only if \( \alpha_n \alpha_{n+1} \alpha_{n+2} \cdots \alpha_{2n-1} \left[ \alpha_{2n}^k - \alpha_{2n}^{2k} \right] e_{2n} = 0. \)
Proof:

\[ T(e_n) = \alpha_n e_{n+1}, \quad T^*(e_n) = \alpha_{n-1} e_{n-1} \]

Using these conditions we can easily compute the following.

\[ T^n(e_n) = \alpha_n \alpha_{n+1} \ldots \alpha_{2n-1} e_{2n} \]

\[ (T^*T)^k(e_n) = \alpha_n^2 e_n \]

\[ T^n(T^*T)^k(e_n) = \alpha_n^{2k+1} \alpha_{n+1} \alpha_{n+2} \ldots \alpha_{2n-1} e_{2n} \]

\[ (T^*T)^kT^n(e_n) = \alpha_n \alpha_{n+1} \alpha_{n+2} \ldots \alpha_{2n-1} \alpha_{2n}^2 e_{2n} \]

Therefore, \( T \in [(n,k)QN] \iff T^n (T^*T)^k(e_n) = (T^*T)^kT^n(e_n) \)

\[ \iff \alpha_n^{2k+1} \alpha_{n+1} \alpha_{n+2} \ldots \alpha_{2n-1} e_{2n} = \alpha_n \alpha_{n+1} \alpha_{n+2} \ldots \alpha_{2n-1} \alpha_{2n}^2 e_{2n} \]

\[ \iff \alpha_{n+1} \alpha_{n+2} \ldots \alpha_{2n-1} [\alpha_n^{2k+1} - \alpha_n \alpha_{2n}^2] e_{2n} = 0. \]

\[ \iff \alpha_n \alpha_{n+1} \alpha_{n+2} \ldots \alpha_{2n-1} [\alpha_n^{2k} - \alpha_{2n}^2] e_{2n} = 0. \]

**Theorem 8.2** A weighted shift \( T^* \) with decreasing weighted sequence \( (\alpha_n) \) is n power k–quasi normal operator if and only if \( \alpha_{n-1} \alpha_{n-2} \ldots \alpha_1 \alpha_0 [\alpha_{n-1}^k - (\alpha_{n-1})^k] e_0 = 0 \)

Proof:

\[ T(e_n) = \alpha_n e_{n+1}, \quad T^*(e_n) = \alpha_{n-1} e_{n-1} \]

Using these conditions we can easily compute the following.

\[ T^n(e_n) = \alpha_{n-1} \alpha_{n-2} \ldots \alpha_1 \alpha_0 e_0 \]

\[ (TT^*)^k(e_n) = \alpha_{n-1}^k e_n \]

\[ T^n(TT^*)^k(e_n) = \alpha_{n-1}^k \alpha_{n-1} \alpha_{n-2} \ldots \alpha_1 \alpha_0 e_0 \]

\[ (TT^*)^kT^n(e_n) = \alpha_{n-1} \alpha_{n-2} \alpha_{n-3} \ldots \alpha_1 \alpha_0 \alpha_{n-1}^k e_0 \]

Therefore, \( T^* \in [(n,k)QN] \iff T^n (T^*T)^k(e_n) = (T^*T)^kT^n(e_n) \)
\[ \alpha_{n-1} \alpha_{n-2} \ldots \alpha_1 \alpha_0 e_0 = \alpha_{n-1} \alpha_{n-2} \ldots \alpha_1 \alpha_0 \alpha_{-1} k e_0 \]
\[ \iff \alpha_{n-1} \alpha_{n-2} \ldots \alpha_1 \alpha_0 \left[ \alpha_{n-1}^k - (\alpha_{-1})^k \right] e_0 = 0 \]

REFERENCES
