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Properties of Orthonormality in 2-fuzzy n-n inner product space

Thangaraj Beaula[1] and Daniel Evans[2]

[1]*Department of Mathematics,T.B.M.L College,*
Porayar - 609307,India
edwinbeaula@yahoo.co.in
[2]*Department of Mathematics,T.B.M.L College,*
Porayar - 609307,India
danielmusic89@gmail.com

Abstract

In this paper new concepts like non-ortho vectors,non stochastic vectors, orthogonality of vectors are introduced.Some properties related to these concepts are developed. Further linear combination of vectors, free, orthonormal basis are defined and theorems involving isometry and orthonormal basis are established.

AMS Subject Classification: 03E72, 46S40

Key Words and Phrases: Non ortho vector, non stochastic vector, linear combination of vectors, orthonormal basis, non stochastic subspace, isometry

1 Introduction

Gahler[7] introduced the theory of 2- norm on a linear space in 1964. In 1984 Katsaras[8] gave the notion of fuzzy norm on a linear space. Further, fuzzy normed spaces were defined in various ways by Cheng and Mordeson[3] and by Bag and Samanta[2]. R.M. Somasundaram and Thangaraj Beaula [10] introduced the notion of fuzzy 2-normed linear space, F(X), N.The concept of 2-inner product space was introduced by C.R. Diminnie, et al[5]. Parijat Sinha, et al introduced the concept of fuzzy 2- inner product space and the notion of ?-2-norm in[9]. The notions of fuzzy inner product space and of fuzzy normed linear space were established in [7] Also, Vijayabalaji and Thillaigovindan [11] introduced the fuzzy n-inner product space as a generalization of the concept of n-inner product space given by Y.J. Cho, et al in[4]. Thangaraj Beaula and Daniel Evans introduced the concept of 2-fuzzy n-n inner product space in[12] as an extension of[11]. Asit Dey and Madhumangal Pal introduced the notion of ortho vector, stochastic fuzzy vectors and ortho-stochastic fuzzy vectors in[1]. This paper extends that to the concepts of non ortho vector and non stochastic vector in the domain of 2-fuzzy n-n inner product spaces and certain important results regarding that are studied.

2 Preliminaries

Definition 1. Let F(X^{n}) be a linear space over a real field. A fuzzy subset N of F(X^{n}) ^{n} R is called 2-fuzzy n-n norm if and only if

(N1) for all $t \in R$, $t \leq 0$,$N(f_1, \dots, f_n, t) = 0$.

(N2) for all $t \in R$, $t \leq 0$,$N(f_1, \dots, f_n, t) = 1$ if and only if f_1,. . .,f_n are linearly dependent.

(N3) for all $t \in R$, $t \leq 0$, $N(f_1, \dots, f_n, t) = 1$ if and only if f_1,. . . ,f_n are linearly dependent.

(N4) for all $t \in R$, $N(f_1, \dots, cf_n, t) = N(f_1, \dots, f_n/|c|)$

(N5) for all $s, t \in R$,$N(f_1, \dots, f_n + f_n^?, s + t) \geq \min N(f_1, \dots, f_n, s), N(f_1, \dots, f_n^?, t)$

(N6) $N(f_1, \dots f_n, t)$ is a non-decreasing function of $t \in R$ and $lim_{t\to\infty} N\,(f_1, \dots, f_n, t)$

The space $(F(X^n)^n, N)$ is called a 2- fuzzy $n - n$ normed linear space.

Definition 2. Let $(F(X^n), \eta)$ be a 2-fuzzy n-n IPS satisfying the condition $\eta(f_1, f_{1,}f_2, \dots, f_n, t_2) > 0$,when $t > 0$ implies that $f_1, f_2, \dots, f_n$ are linearly dependent. Then for all $(0, 1)$,define$\| \; f_1 \dots f_n \; \|_\infty = inf\{t : \eta(f_1, f_1, f_2, \dots, f_n, t^2) \geq \alpha\}$ a crisp norm on $F(X^n)$ called the $\alpha - n - n$ norm and the space is$(F(x^n), \|\|_\infty)$ generated by η.

3 Non -ortho vectors and non- stochastic vectors

Definition 3. A fuzzy vector $f = (f_1, \dots, f_n)$ in $[F\,(X^n)]^n$ is said to be non-ortho vector if $f_i \cdot f_j = 0$ for $i, j = 1, 2, \dots, n$.

Definition 4. A fuzzy vector $f = (f_1, \dots, f_n)$ is said to be a non-stochastic vector if $min\,(f_i) = 0$.

Definition 5. Let $f = (f_1, \dots, f_n)\,, g = (g_1, \dots, g_n) \in [F\,(X^n)]^n$ then the inner product of f,g is denoted by $\langle f, g\rangle$ and is defined by
$\langle f, g\rangle = min_{i=1,\dots,n}\,(max\,(f_i\,(x)\,, g_i\,(x)))$

Definition 6. Let $f = (f_1, \dots, f_n) \in [F\,(X^n)]^n$. The norm of f is denoted by $\|f\|$ and $\|f\| = \langle f, f\rangle$ the vector is said to be a unit vector if $\|f\| = 1$.

Definition 7. Let $f, g, h \in [F\,(X^n)]^n$ and $\alpha \in R$,then

(i) $\langle \alpha f + g, h\rangle = \alpha\langle f, h\rangle + \langle g, h\rangle$

(ii) $\langle f, g \rangle = \langle g, h \rangle$

(iii) $\langle \alpha f, g \rangle = \langle f, \alpha g \rangle = \alpha \langle f, g \rangle$

(iv) $\langle f, f \rangle = 0$ if and only if $f = (0, \ldots, 0)$

Definition 8. Two fuzzy vectors f,g in $[F(X^n)]^n$ are said to be orthogonal if $\langle f, g \rangle = 0$. In this case we write $f \perp g$.

Theorem 9. *Let $f, g \in [F(X^n)]^n$ and $\alpha \in R$ then*

(i) $\|cf\| = c\langle cf, f \rangle$

(ii) $\|f + g\| = max\ (\|f\|, \|g\|)$

(iii) $\langle f, g \rangle \geq max\ (\|f\|, \|g\|)$

Proof. (i) $\|cf\| = \langle cf, cf \rangle = c\langle f, cf \rangle = c\langle cf, f \rangle$

$$\begin{aligned}(ii)(min_{i=1,\ldots,n}(max(f_i + g_i)) &= min\ (max\ (max\ (f_i, g_i), max\ (f_i, g_i))) \\ &= min\ (max\ (f_i, g_i)) \\ &= max\ (min\ (f_i), min\ (g_i)) \\ &= max\ (\|f\|, \|g\|)\end{aligned}$$

$$\begin{aligned}(\text{ iii})\langle f, g \rangle = \bigwedge_{i=1}^{n} (f_i \bigvee\ g_i) \geq \bigwedge_{i,j=1}^{n} (f_i \bigvee\ g_i) &= \left(\bigwedge_{i=1}^{n} (f_i)\right) \vee \left(?_{j=1}^{n} (g_i)\right) \\ &= max\ (\|f\|, \|g\|).\end{aligned}$$ □

Theorem 10. *If f and g are two non stochastic vectors in $[F(X^n)]^n$ then $\langle f, g \rangle = 0$ if $f = g$ but the converse need not be true*

Proof. Let $f = g$ then $f_i = g_i$ for $i = 1, 2, \ldots, n$
Since f,g are non stochastic then $f_i = 0,\ \ g_i = 0$ for some i,
Therefore $\langle f, g \rangle = \bigwedge_{i=1}^{n} (f_i \bigvee g_i) = 0$
But if $f = (0, \ldots, 0)$ and $g = (0, \ldots, 0)$, then both f and g are non stochastic and $\langle f, g \rangle = 0$, but $f \neq g$. Hence the converse need not be true □

Theorem 11. *Let $f, g \in [F(X^n)]^n$ and $\alpha \in R$,then*

(i) $\|cf\| = c\|f\|$

(ii) $\|f + g\| = \|f\| \bigvee \|g\|$

(iii) $\langle f, g \rangle \geq \|f\| \bigvee \|g\|$

Proof. It is obvious. □

Definition 12. A subset E of $[F(X^n)]^n$ is said to be a basis of $[F(X^n)]^n$ if E is generating and free.

Theorem 13. *Let E be a basis of $[F(X^n)]^n$, if $f \in E$ then $\|f\| = 1$*

Proof. Let $(0, 0, \ldots, 0) \in E$. Then $1 = 1\,(1, \ldots, 1)\,, 1 = 1 + 0$. Then 1 can be written as two distinct linear combination of element of E. This is a contradiction because E is free. Thus $0 \notin E$.

If possible let $f \in E$ with $\|f\| \neq 1$ .Then $f = 1 \cdot f$ and $f = \|f\|f$, because $\|f\| = \bigwedge_{i=1}^{n} f_i$

$$\begin{aligned} \|f\|f &= \left(\bigwedge_{i=1}^{n} f_i\right) f \\ &= ((?_{i=1}^{n} f_i)?\ f_1, (?_{i=1}^{n} f_i)?f_2, \ldots, (?_{i=1}^{n} f_i)?f_n) \\ &= (f_1, \ldots, f_n) \\ &= f \end{aligned}$$

In this case f can be written as two distinct linear combination of elements of E with non zero coefficient. This is a contradiction because E is a basis.
Therefore $\|f\| \neq 1$ is wrong, hence $\|f\| = 1$. □

Theorem 14. *Let E be an orthonormal set* $[F\,(X^n)]^n$ *then E is free.*

Proof. Let $f \in [F\,(X^n)]^n$ with f $= \sum_{i=1}^{m} \alpha_i f_i = \sum_{i=1}^{k} \beta_i g_i$, where $f_i, g_i \in E$ and $\alpha_i, \beta_i \in R - \{0\}$
$\beta_i = \langle f, g_i \rangle$ for $i = 1, 2, \ldots, k$, if $g_j \notin \{f_1, \ldots, f_n\}$ for some j then

$$\beta_j = \langle g_j, f \rangle = \langle g_j, \sum_{i=1}^{m} \alpha_i f_i \rangle = 0$$

which is a contradiction, since $\beta_j \in R - \{0\}$
Thus $g_j \in \{f_1, \ldots, f_m\} \Rightarrow \{g_1, \ldots, g_k\} \subseteq \{f_1, \ldots, f_m\}$
Similarly, $\alpha_i = \langle f, f_i \rangle$ for $i = 1, 2, \ldots m$. If $f_j \notin \{g_1, \ldots, g_k\}$ for some j.

Then $f_j = \langle f_j, f \rangle = \beta_j = \langle g_j, f \rangle = \langle f_j, \sum_{i=1}^{m} \beta_j g_i \rangle = 0$,
which is a contradiction since $\alpha_i \in R - \{0\}$,$f_j \in \{g_1, \ldots, g_k\}$,
$\Rightarrow \{f_1, \ldots, f_m\} \subseteq \{g_1, \ldots, g_k\}$, $\Rightarrow \{f_1, \ldots, f_m\} = \{g_1, \ldots, g_k\}$,$\Rightarrow m = k$

Now $\alpha_i = \langle f, f_i \rangle = \langle f, g_j \rangle = \beta_j$ for all $i \in \{1, 2, \ldots, m\}$,$j \in \{1, \ldots, k\}$ and so E is free. □

4 Properties of Linear Transformation and Isometry

Definition 15. Let $G, H \subseteq [F\,(X^n)]^n$ be two subspaces and T be a linear map. Then T is said to be an isomorphism if T is a surjective isometry. Here G,H are called isomorphism subspaces.

Theorem 16. *Let* $G, H \subseteq [F\,(X^n)]^n$ *be two subspaces and* $T : G \to H$ *be an isomorphism. Then T maps orthonormal bases to orthonormal bases.*

Proof. Since T is an isomorphism, T is a surjective isometry. Let $\{f_1, \ldots, f_n\}$ be an orthonormal basis of G then $\{T(f_1), \ldots, T(f_n)\} \subseteq H$. Thus $L\{T(f_1), \ldots, T(f_n)\} \subseteq H$. Let g be an arbitrary element of H.
Since T is surjective, there exists $f \in G$ such that $T(f) = g$. Since $f \in G$ and $\{f_1, \ldots, f_n\}$ is an orthonormal basis of G. Therefore $\alpha_1, \ldots, \alpha_n \in R$ such that $f = \sum_{i=1}^{n} \alpha_i f_i$.

Now $g = T(f) = T(\sum_{i=1}^{n} \alpha_i f_i) = \sum_{i=1}^{n} \alpha_i T(f_i) \in L\{T(f_1), \ldots, T(f_n)\}$ and $H = \{T(f_1), \ldots, T(f_n)\}$
Thus $\{T(f_1), \ldots, T(f_n)\}$ is a generating subset of H. Since T is an isometry and $\{f_1, \ldots, f_n\}$ is an orthonormal basis. Thus $\{T(f_1), \ldots, T(f_n)\}$ is an orthonormal generating subset of H.
Hence $\{T(f_1), \ldots, T(f_n)\}$ is a orthonormal basis of H. □

Definition 17. A linear map $T : G \to H$ between two subspaces G,H of $[F(X^n)]^n$ is called is an isometry when for all $f, g \in G$, we have $\langle T(f), T(g)\rangle = \langle f, g\rangle$.

Theorem 18. *Let $G, H \subseteq [F(X^n)]^n$ be two subspaces and $T : G \to H$ be a linear map. Then the following are equivalent.*

(i) For any orthonormal set $A = \{\delta_1, \ldots, \delta_k\}$ of G, the set $\{T(\delta_1), \ldots, T(\delta_k)\}$ is an orthonormal set of H.

(ii) There exists an orthonormal basis $A = \{\delta_1, \ldots, \delta_k\}$ of a such that $\{T(\delta_1), \ldots, T(\delta_k)\}$ is an orthonormal set of H.

(iii) T is an isometry.

Proof. (i) $\Longrightarrow$(ii) Since $A = \{\delta_1, \ldots, \delta_k\}$ is the orthogonal basis of A, so A is an orthonormal set of H. Thus by (i), $\{T(\delta_1), \ldots, T(\delta_k)\}$ is an orthogonal set of H.

(ii)$\Longrightarrow$(iii) Let $A = \{\delta_1, \ldots, \delta_k\}$ be a basis of G such that $\{T(\delta_1), \ldots, T(\delta_k)\}$ is an orthonormal set of H. Let $f.g \in G$. Then $f = |\sum_{i=1}^{k} f_i\delta_i|$,$g = |\sum_{i=1}^{k} g_i\delta_i|$. Then $f = \sum_{i=1}^{k} \alpha_i\delta_i$, $g = \sum_{i=1}^{k} \beta_i\delta_i$, where $\alpha_i, \beta_i \in R$.

$$\text{Now } \langle T(f), T(g)\rangle = \langle \textstyle\sum_{i=1}^{k} \alpha_i T(\delta_i), \sum_{j=1}^{k} \beta_j T(\delta_j)\rangle = \bigwedge_{i,j=1}^{k} \alpha_i\beta_j \langle T(\delta_i), T(\delta_j)\rangle$$
$$= \bigwedge_{i,=1}^{k} \alpha_i\beta_j \langle T(\delta_i), T(\delta_i)\rangle = \bigwedge_{i,=1}^{k} \alpha_i\beta_j = \langle f, g\rangle).$$

Because $\{T(\delta_1), \ldots, T(\delta_k)\}$ is an orthonormal set of $[F(X^n)]^n$ .Thus T is an isometry
(iii) $\Longrightarrow$ (i) Assume $\langle T(f), T(g)\rangle = \langle f, g\rangle$ for all $f, g \in G$ Now $\langle$ T $(\delta_i), T(\delta_j)\rangle =$
$\langle \delta_i, \delta_j\rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$
Hence $\{T(\delta_1), \ldots, T(\delta_k)\}$ is an orthonormal set. □

Definition 19. Let $[F(X^n)]^n$ be a 2-fuzzy n-n inner product space and $f \in [F(X^n)]^n$. Then the orthogonal complement of f is denoted by $f^{\perp}$ and is defined by $f^{\perp} = \{g \in [F(X^n)]^n : \langle f, g\rangle = 0\}$. Let G be a subspace of $F[(X^n)]^n$. Then the orthogonal complement of G is denoted by $G^{\perp}$ and is defined by $f^{\perp} = \{g \in [F(X^n)]^n : \langle f, g\rangle = 0$ for all $f \in G\}$

Theorem 20. *Let $M \subseteq [F(X^n)]^n$ and $M^{\perp}$ be the orthogonal complement of M.Then the following hold*

(i) *$M^{\perp}$ is a subspace of $[F(X^n)]^n, M \subseteq M^{\perp\perp}$ and $M \cap M^{\perp} = \{0\}$.*

(ii) *Let $N \subseteq [F(X^n)]^n$ with $M \subseteq N$,then $N^{\perp} \subseteq M^{\perp}$.*

(iii) *$\{0\}^{\perp} = [F(X^n)]^n$ and $([F(X^n)]^n)^{\perp} = \{0\}$ and $M^{\perp} = M^{\perp\perp\perp}$.*

(iv) *Let $M, N \subseteq F[(X^n)]^n$,then $(M+N)^{\perp} = M^{\perp} \cap N^{\perp}$,where $M + N = \{f + g : f \in M, g \in N\}$.*

Proof. (i) Let $f, g \in M^{\perp}, \alpha, \beta \in R$ for any $h \in M$ $\langle \alpha f + \beta g, c\rangle = \alpha\langle f, c\rangle + \beta\langle g, c\rangle = 0$
$\Rightarrow \alpha f + \beta g \in M^{\perp}$. Thus $M^{\perp}$ is a subspace of $F[(X^n)]^n$

Let $f \in M$. Then $\langle f, g\rangle = 0$ for all $g \in M^{\perp}$
$\Rightarrow f \in (M^{\perp})^{\perp} = M^{\perp\perp} \Rightarrow M \subseteq M^{\perp\perp}$

Again $f \in M \cap M^{\perp} \Rightarrow \langle f,f\rangle = 0 \Rightarrow f = 0$

Hence $M \cap M^{\perp} = \{0\}$. (ii) Let $f \in N^{\perp} \Rightarrow \langle f, g\rangle = 0$ for all $g \in N \Rightarrow \langle f, g\rangle = 0$ for all $b \in M \subseteq N$
$\Rightarrow f \in M^{\perp}$ therefore $N^{\perp} \subseteq M^{\perp}$
(iii) $\{0\}^{\perp} = \{f \in F[(X^n)]^n : \langle 0, f\rangle = 0\} = F[(X^n)]^n$

also when $f \neq 0$ then $\langle f, f\rangle \neq 0$
(i.e) a non zero element cannot be orthogonal to the entire space.
Therefore,$([F(X^n)]^n)^{\perp} = \{0\}$
Let $f \in M$, then $\langle f, g\rangle = 0$ for all $g \in M^{\perp} \Rightarrow f \in M^{\perp\perp}$

Then $M \subseteq M^{\perp\perp}$. Changing M by $M^{\perp}$,$M^{\perp} \subseteq \left(\left(M^{\perp}\right)^{\perp}\right)^{\perp} = M^{\perp\perp\perp}$.

Also , $M \subseteq M^{\perp\perp} \Rightarrow (M^{\perp\perp})^{\perp} \subseteq M^{\perp} \Rightarrow M^{\perp\perp\perp} \subseteq M^{\perp}$
therefore $M^{\perp} = M^{\perp\perp\perp}$
(iv) Let $f \in (M+N)^{\perp}$ then $\langle f, g\rangle = 0$ for all $g \in M + N$
Let $g = g_1 + g_2$ where $g_1 \in M$ and $g_2 \in N$

Consider $\langle f, g\rangle = 0 \Rightarrow \langle f, g_1 + g_2\rangle = 0 \Rightarrow \langle f, g_1\rangle \vee \langle f, g_2\rangle = 0 \Rightarrow \langle f, g_1\rangle = 0$ and $\langle f, g_2\rangle = 0$ for all $g_1 \in M$, $g_2 \in N$. Therefore $f \in M^{\perp}$ and $f \in N^{\perp}$ and so $f \in M^{\perp} \cap N^{\perp}$
therefore $(M+N)^{\perp} \subseteq M^{\perp} \cap N^{\perp}$

Conversely, let $f \in M^{\perp} \cap N^{\perp} \Rightarrow f \in M^{\perp}$ and $f \in N^{\perp} \Rightarrow \langle f, g\rangle = 0$ for all $g \in M$ and $\langle f, h\rangle = 0$ for all $h \in N$

Let $k \in M + N \Rightarrow k = g + h$, where $g \in M, h \in N$

Consider $\langle f, k\rangle = \langle f, g + h\rangle = \langle f, g\rangle \bigvee \langle f, h\rangle = 0$ for all K. Therefore $f \in (M + N)^{\perp}$ and so $f \in M^{\perp} \cap N^{\perp} \subseteq (M + N)^{\perp} \Rightarrow (M + N)^{\perp} = M^{\perp} \cap N^{\perp}$

□

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