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Finite Dimensional Fuzzy BI- Normed Linear Space

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Abstract

In this paper, the notion of fuzzy Bi-normed linear spaces is introduced standard results in Bi-normed linear spaces are extended to fuzzy Bi-normed linear spaces. It is proved that in a finite dimensional fuzzy Bi-normed linear space, fuzzy Bi-norms are the same upto fuzzy equivalence, finite dimensional fuzzy subspaces of a fuzzy bi-normed linear space are discussed. Certain results in connection with the Bi-normed convergent sequences are also analyzed.

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1 Introduction

The concept of fuzzy set was introduced by Zadeh[7] in 1965. S.Gahler[3], introduced the Lineare 2-normierte Raume. C.Felbin[2] introduced the definition of Finite dimensional fuzzy normed linear spaces. In this paper, we use definition of fuuy point in the linear space of fuzzy Bi-norms $P^*(X)$ as mentioned by Hu Cheng-ming[3]. Meenakshi A.R. and Cokilavany R. [5] introduced the concept of fuzzy 2-normed linear spaces. A.Nagoorgani and G.Kalyani[6] introduced the 2-normed sequence in matrices. The organization of the paper is as follows: Section 2, provides some preliminary results which are used in this paper. In section 3, we have discussed the fuzzy equivalence of fuzzy Bi-norm, then triangle inequality theorem is proved and certain results about this convergence and F- Cauchy sequence theorem is proved in the realm of fuzzy bi-normed linear spaces in $P^*(X)$ in Section 4. Section 5, a finite dimensional F-subspace of any fuzzy Bi-normed linear space is necessarily a complete fuzzy Bi-normed linear space.

2 Preliminaries

Let X be a nonempty set, $I = [0, 1]$ be the unit interval. The pair $(x, \alpha), x \in X, \alpha \in I$ is called a fuzzy point in X, denoted by P_x^α or $P(x, \alpha)$, some times simply P Fuzzy point $P^{(1-\alpha)}$ is called the dual point of P_x^α usually the dual point of P is denoted by $P^*(X)$. The set of all fuzzy points in X is denoted by $P^*(X) = P_x^\alpha x \in X, \alpha \in [0, 1]$. For any fuzzy set $A \in I^x$, the collection of all mappings of X into I. we say that $P_x^\alpha \in A \implies \alpha < A(x)$ or $A(x) = 1$ and $P_x^\alpha \in A \implies \alpha \leq (x)$and $A(x) \neq 0$.
Let $P_x^\alpha, P_y^\beta \in P^*(X)$ be fuzzy points, then we say that $P_x^\alpha \leq P_y^\beta \implies x = y, \alpha \leq \beta$ and $P_x^\alpha < P_y^\beta \implies = y, \alpha < \beta$.

3 Fuzzy equivalence of fuzzy Bi-norm

Definition 1. A fuzzy bi-norm for a real or complex linear space X is a real valued function θ defined on $P^*(X) \times P^*(X)$ to $[0, 1]$ satisfying the following conditions. For any $P_1, P_2, P_3 \in P^*(X)$ and $\alpha \in [0, 1]$
$(i)\theta(P_1, P_2) = 0$ if and only if $P_1 \leq P_2$ or $P_2 \leq P_1$
$(ii)\theta(P_1, P_2) = \theta(P_2^*, P_1^*)$
$(iii)\theta(rP_1, P_2) = |r|\theta(P_1, P_2), \forall \in [0, 1]$
$(iv)\forall P_1, P_2, P_3 \in P^*(X)$ and $r_1, r_2, r_3 \in [0, 1]$
$(a)\theta(P_1 + P_2, P_3)(r_1 + r_2, r_3) \geq L(\theta(P_1, P_3)(r_1, r_3) + \theta(P_2, P_3)(r_2, r_3))$
Where $(r_1, r_2) \leq \theta(P_1, P_3), (r_2, r_3) \leq \theta(P_2, P_3)$ and $(r_1 + r_2, r_3) \leq \theta(P_1 + P_2, P_3)$
$(b)\theta(P_1 + P_2, P_3)(r_1 + r_2, r_3) \leq R(\theta(P_1, P_3)(r_1, r_3) + \theta(P_2, P_3)(r_2, r_3))$
Where $(r_1, r_2) \geq \theta(P_1, P_3), (r_2, r_3) \geq \theta(P_2, P_3)$ and $(r_1 + r_2, r_3) \geq \theta(P_1 + P_2, P_3)$
and (X, θ) is called a fuzzy Bi-normed linear space. If (i) is omitted then a is called fuzzy pseudo-Bi-norm and (X, θ) is called a fuzzy pseudo-Bi-normed linear space. We take $L(P_1, P_2) = min(P_1, P_2)$, $R(P_1, P_2) = Max(P_1, P_2)$ $[L, R : [0, 1] \times [0, 1] \to [0, 1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1]$ and we write (X, θ) or simply X when L and R are as indicated just above the following result is the analogue of the usual triangle inequality.
Then (X, θ) is called a fuzzy bi-normed linear space with respect to the Bi-norm. $[\theta(P_1, P_2)$ is defined for $P_1 = P_2$ as 0 (by (i)). Therefore $\theta(P_1, P_2) \neq 0$ is defined for $P_1 < P_2]$
(i) When $P_1 = P_2$ both the fuzzy points coincide.
Therefore $\theta(P_1, P_2) = 0$. Conversely $\theta(P_1, P_2) = 0 \implies$ both fuzzy points should coincide $P_1 = P_2$
$(ii)P_1 < P_2 \implies r_1 < r_2 \implies 1 - r_2 < 1 - r_1 \implies P_2^* < P_1^*$.

Therefore $\theta(P_1, P_2) = \theta(P_2^*, P_1^*)$
$(iii)Case(i)$ Let $r < r_1$

$$\theta(rP_1, P_2) \implies rP_1 < P_2 \implies rr_1 < r_2 \implies r < r_2(since r < r_1 < r_2) \quad (1)$$
$$r\theta(P_1, P_2) \implies rr_1 < rr_2 \implies < r \implies < r_2(since r < r_1 < r_2) \quad (2)$$

From (1) and (2)
$\theta(rP_1, P_2) \implies (P_1, P_2)$
Case(ii) Let $r_1 < r$

$$\theta(rP_1, P_2) \implies rP_1 < P_2 \implies rr_1 < r_2 \implies r < r_2(since r < r_1 < r_2) \quad (3)$$
$$r\theta(P_1, P_2) \implies rr_1 < rr_2 \implies r_1 < r_2 if r_2 < r(since r < r_1 < r_2) \quad (4)$$

From (3) and (4) $\theta(rP_1, P_2) \implies r\theta(P_1, P_2)$
Case(i) Let $r_2 < r_1$
$P_1 + P_2 < P_3 \implies r_1 + r_2 < r_3 \implies r_1 < r_3 \implies P_1 < P_3$

$$\theta(P_1 + P_2, P_3) = \theta(P_1, P_3) \quad (5)$$

Case(ii) Let $r_1 < r_2$
$P_1 + P_2 < P_3 \implies r_1 + r_2 < r_3 \implies r_2 < r_3 \implies P_2 < P_3$

$$\theta(P_1 + P_2, P_3) = \theta(P_2, P_3) \quad (6)$$

From (5) and (6)
$\theta(P_1 + P_2, P_3) = either \theta(P_1, P_3) or \theta(P_2, P_3)$
Therefore $\theta(P_1 + P_2, P_3) \le \theta(P_1, P_3) + \theta(P_2, P_3)$
Since $\theta(P_2, P_3) \ge 0$ [infact$\theta(P_1 + P_2, P_3) = \theta(P_1, P_3) + \theta(P_2, P_3)$]
Thus θ is a fuzzy bi-norm with respect to the Bi-norm in $P^*(X)$ and hence $P^*(X)$ is a fuzzy bi-normed linear space with respect to the Bi-norm.

Theorem 2. *In a fuzzy Bi-normed linear space (X, θ) the triangle inequality of definition(2) is equivalent to $\theta(P_1 + P_2, P_3)\theta(P_1, P_3) + \theta(P_2, P_3)$*

Proof. By definition(2) (iv), case(a) and case(b) □

Theorem 3. *The triangle inequality in definition(2) (iv) (b) (with R=Max) is equivalent to the triangle inequality $\theta_2^\alpha(P_1 + P_2, P_3) \le \theta_2^\alpha(P_1, P_3) + \theta_2^\alpha(P_2, P_3)$ for all $\alpha \in [0, 1]$ and $P_1, P_2, P_3 \in P^*(X)$*

Proof. Let def1 hold and let $P_1, P_2, P_3 \in P^*(X)$, $(r_1, r_2) \le \theta(P_1, P_3)$ and $(r_2, r_3) \ge \theta(P_2, P_3)$ Denote $\alpha = \theta(P_1 + P_2, P_3)(r_1 + r_2, r_3)$ then $(r_1 + r_2, r_3) \le \theta_2(P_1 + P_2, P_3)$, $(r_1 + r_2, r_3) \le \theta_2(P_1, P_3) + \theta_2(P_2, P_3)$. Hence $(r_1, r_3) \le \theta_2^\alpha(P_1, P_3)$ or $(r_2, r_3) \le \theta_2^\alpha(P_2, P_3)$. $\implies$ $\theta(P_1, P_3)(r_1, r_3) \ge \alpha$ or $\theta(P_2, P_3)(r_2, r_3) \ge \alpha \implies Max(\theta(P_1, P_3)(r_1, r_3), \theta(P_2, P_3)(r_2, r_3)) \ge \alpha = \theta(P_1 + P_2, P_3)(r_1 + r_2, r_3)$.
Conversely,

Assume that definition (2)(b) holds and let $P_1, P_2, P_3 \in P^*(X)$ and $\alpha \in [0,1]$ Suppose $\theta_2^\alpha(P_1 + P_2, P_3) > \theta_2^\alpha(P_1, P_3) + \theta_2^\alpha(P_2, P_3)$, then there exist $(r_1, r_3) > \theta_2^\alpha(P_1, P_3) \geq \theta_1^1(P_1, P_3)$ and $(r_2, r_3) > \theta_2^\alpha(P_2, P_3) \geq \theta_2^\alpha(P_2, P_3)$ such that $(r_1 + r_2, r_3) = \theta_2^\alpha(P_1 + P_2, P_3) \geq \theta_1^1(P_1 + P_2, P_3)$. Hence by definition(2) (iv)(b), $\alpha = \theta(P_1 + P_2, P_3)(r_1 + r_2, r_3) \leq Max(\theta(P_1, P_3)(r_1, r_3), \theta(P_2, P_3)(r_2, r_3))$ This contradiction proves the inequality of the theorem. □

Theorem 4. *The triangle inequality in definition (2) (iv)(a) (with L=min) is equivalent to the triangle inequality*

$$\theta_1^\alpha(P_1 + P_2, P_3) \leq \theta_1^\alpha(P_1, P_3) + \theta_1^\alpha(P_2, P_3) \tag{7}$$

for all$\alpha \in [0,1]$ *and* $P_1, P_2, P_3 \in P^*(X)$

Proof. Assume that definition(2) holds and let $P_1, P_2, P_3 \in P^*(X), \forall \alpha, \beta, \gamma \in [0,1]$ $(r_1, r_2) \leq \theta(P_1, P_3)$ and $(r_2, r_3) \leq \theta(P_2, P_3)$ Denote $\alpha = \theta(P_1, P_3)(r_1, r_3)$ and $\beta = \theta(P_2, P_3)(r_2, r_3)$ then $(r_1 + r_2, r_3) \leq \theta_1^1(P_1 + P_2, P_3)$. $\gamma = min(\alpha, \beta)$. Now $\theta_1^\alpha(P_1, P_3) \leq (r_1, r_3), \theta_1^\beta(P_2, P_3) \leq (r_2, r_3)$. Since $\|.\|_1^\alpha$ or θ_1^α is non- decreasing in α. We have by theorem1. $\theta_1^\gamma(P_1 + P_2, P_3) \leq \theta_1^\gamma(P_1, P_3) + \theta_1^\gamma(P_2, P_3) \leq \theta_1^\alpha(P_1, P_3) + \theta_1^\beta(P_2, P_3) \leq (r_1, r_3) + (r_2, r_3) \leq \theta_1^1(P_1 + P_2, P_3)$.
$\implies \theta(P_1 + P_2, P_3)(r_1 + r_2, r_3) \geq \gamma = min(\theta(P_1, P_3)(r_1, r_3), \theta(P_2, P_3)(r_2, r_3))$
Hence $(r_1, r_3) \leq \theta_2^\alpha(P_1, P_3)$ or $(r_2, r_3) \leq \theta_2^\alpha(P_2, P_3)$. $\implies \theta(P_1, P_3)(r_1, r_3) \geq \alpha$ or $\theta(P_2, P_3)(r_2, r_3) \geq \alpha$. $\implies Max(\theta(P_1, P_3)(r_1, r_3), \theta(P_2, P_3)(r_2, r_3)) \geq \alpha = \theta(P_1 + P_2, P_3)(r_1 + r_2, r_3)$.
Conversely,
Assume that definition(2)(a) holds with L=min and let $P_1, P_2, P_3 \in P^*(X)$ and $\alpha \in [0,1]$. Suppose $\theta(P_1 + P_2, P_3)(\theta_1^\alpha(P_1, P_3) + \theta_1^\alpha(P_2, P_3)) = min((\theta(P_1, P_3)\theta_1^\alpha(P_1, P_3)), (\theta(P_2, P_3)\theta_1^\alpha(P_2, P_3))) = \alpha$. If $\theta_1^\alpha(P_1, P_3) + \theta_1^\alpha(P_2, P_3) \leq \theta_1^1(P_1 + P_2, P_3) \implies \theta_1^\gamma(P_1 + P_2, P_3) \leq \theta_1^\gamma(P_1, P_3) + \theta_1^\gamma(P_2, P_3)$ If on the other hand, $\theta_1^\gamma(P_1 + P_2, P_3) \leq \theta_1^\gamma(P_1, P_3) + \theta_1^\gamma(P_2, P_3)$ Therefore θ_1^α is non- decreasing in α □

Example 5. *If* $P^*(X) = R^n$ *and* θ *is defined by*

$$\theta(P_1, P_2, , P_n)(r) = \begin{cases} 1 & \text{if } r = \sqrt{(P_1^2 + P_2^2 + \; + P_n^2)} \\ 0 & \text{if otherwise} \end{cases}$$

For $(P_1, P_2, , P_n) \in P^*(X)$, *then* θ *is a fuzzy Bi-norm according to definition3.1 with the choice of L and R indicated above.*

4 Convergence and F-cauchy sequence for fuzzy Bi-normed linear spaces

Definition 6. Let (X, θ) be a fuzzy Bi-normed linear space. A sequence $\{P_{(x_n)}\} \in P^*(X)$ is said to converge to $P_x \in P^*(X)$ denoted by $\lim_{n\to\infty} P_{(x_n)} = P_x$ if and only if $\lim_{n\to\infty} \theta(P_{(x_n)} - P_x, P_y)$.
i.e. $\lim_{n\to\infty} \theta_1^\alpha(P_{(x_n)} - P_x, P_y) = \lim_{n\to\infty} \theta(P_{(x_n)} - P_x, P_y) = 0$ for $\alpha \in [0,1]$

Definition 7. A sequence $\{P_{(x_n)}\}$ is a fuzzy Bi-normed linear space (X, θ) is called F-Cauchy sequence if there exists $P_x, P_y, P_z \in P^*(X)$ such that P_y and P_z

are linearly independent, the $\lim_{n\to\infty} \theta(P_{(x_m)} - P_{(x_n)}, P_y) = 0$ and the $\lim_{n\to\infty} \theta(P_{(x_n)} - P_{(x_m)}, P_z) = 0$.
i.e. $\lim_{n\to\infty} \theta_2^{\alpha}(P_{(x_m)} - P_{(x_n)}, P_y) = 0$ and $\lim_{n\to\infty} \theta_2^{\alpha}(P_{(x_n)} - P_{(x_m)}, P_z) = 0$ for $\alpha \in [0,1]$

Theorem 8. *All subsequences of a convergent sequence converge to the limit of the sequence in a fuzzy Bi-normed linear space (X, θ).*

Proof. Let $\{P_{(x_n)}\}$ converge to P_x. Then $\lim_{n\to\infty} \theta_2^{\alpha}(P_{(x_m)} - P_{(x_n)}, P_y) = 0$ for $\alpha \in [0,1]$
i.e. given $\varepsilon > 0$ for any fixed $\alpha \in [0,1]$ there exists N such that for every $n \geq (\alpha)$ $\theta_2^{\alpha}(P_{(x_n)} - P_x, P_y) < \varepsilon$. Let $\{P_(x_(k(n)))\}$ be a subsequence of $\{x_n\}$ such that $k(n) < k(m)$ for $n < m$.
Since $\{P_{(x_{(k(n))})}\}$ is a subsequence there exists $M(\alpha)$ such that $N(n) \leq N(\alpha)$ for $n \geq M(\alpha)$.
Therefore $n \geq M(\alpha) \implies \theta_2^{\alpha}(P_(x_(k(n))) - P_x, P_y) < \varepsilon$
i.e. $\lim_{n\to\infty} \theta(P_{(x_{k(n)})} - P_x, P_y) = 0$ or $\{P_(x_(k(n)))\}$ converges to $P_x \in P^*(X)$.
□

Theorem 9. *In a fuzzy Bi-normed linear space (X, θ) every convergent sequence is also a Cauchy sequence.*

Proof. Let $\{P_{(x_n)}\}$ converge to $P_x \in P^*(X)$. Then $\lim_{n\to\infty} \theta_2^{\alpha}(P_{(x_n)} - P_x, P_y) = 0$ for $\alpha \in [0,1]$
i.e. given $\varepsilon > 0$ for any fixed $\alpha \in [0,1]$ there exists N_0 such that for every $n \geq N_0()$ $\theta_2^{\alpha}(P_{(x_n)} - P_x, P_y) \leq \frac{\varepsilon}{2}$ Now θ_2^{α} being a Bi-normed linear space. $\theta_2^{\alpha}(P_{(x_m)} - P_{(x_n)}, P_y) = \theta_2^{\alpha}((P_{(x_m)} - P_x, P_y) - (P_{(x_n)} + P_x, P_y)) = \theta_2^{\alpha}(P_{(x_m)} - P_x, P_y) + \theta_2^{\alpha}(P_{(x_n)} - P_x, P_y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$
For every ,$n \geq N_0(\alpha)$. Therefore fuzzy Bi-normed linear space (X, θ) every convergent sequence is also a Cauchy sequence □

5 Fuzzy completeness of finite dimensional F-subspace

Definition 10. Let (X, θ) be a fuzzy Bi-normed linear space, Y is called a fuzzy subspace, if Y is a subspace of X considered as a vector space with the fuzzy Bi-norm obtained by restricting the fuzzy Bi-norm on X to the subset Y the fuzzy Bi-norm on Y is said to be induced by the fuzzy Bi-norm on X.

Definition 11. A fuzzy Bi-normed linear space in which every F-cauchy sequence converges is called a complete fuzzy Bi-normed linear space

Theorem 12. *Every finite dimensional F-subspace Y of a fuzzy Bi-normed linear space X is complete. In particular every finite dimensional fuzzy Bi-normed linear space is complete.*

Proof. let $dimY = n$ and $\{e_1, e_2, , e_n\}$ be any basis for Y. let $\{P_(y_m)\}$ be any arbitrary F-cauchy sequence in Y, then each $\{P_(y_m)\}$ has a unique representation of the form $P_(y_m) = a_1^m e_1 + a_2^m e_2 + \; + a_m^m e_n$ and $\lim_{n\to\infty} \theta(P_(y_m) - P_(y_(r)), P_z) = 0$

i.e. $\lim_{n\to\infty} \theta_2^\alpha(P_(y_m) - P_(y_(r)), P_z) = 0$ for $\alpha \in [0,1]$.
i.e. given $\varepsilon > 0$ and $\alpha \in [0,1]$ there exists N such that $\theta_2^\alpha(P_(y_m) - P_(y_(r)), P_z) \leq \varepsilon$ Show that each of the n sequences $a_j^m = a_j^1, a_j^2, a_j^m, j = 1, 2, , n$ is Cauchy in $P^*(X)$ and hence each converges to $a_j, j = 1, 2, , n$.Define $P_y = a_1e_1 + a_2e_2 + +a_ne_n$ For every $\alpha \leq \alpha_0 \theta_2^\alpha(P_(y_m) - P_y, P_z) = \theta_2^\alpha((a_1^m - a_1)e_1 + +(a_n^m - a_n)e_n)$ $\theta_2^\alpha(P_(y_m) - P_y, P_z) \leq |a_j^m - a_1|\theta_2^\alpha(e_1) + +|a_n^m - a_n|\theta_2^\alpha(e_n)$ Since $\theta_2^\alpha(e_j), j = 1, 2, , n$ are finite and $a_j^m \to a_j$ $\implies \theta_2^\alpha(P_(y_m) - P_y, P_z) \to 0$ as $m \to \infty$ and $\theta(P_(y_m) - P_y, P_z) \to 0$ as $m \to \infty$. Therefore $\{P_(y_n)\}$ converge to P_y □

6 Conclusion

In this paper, we considered the fuzzy Bi-normed linear space and some basic concepts. Convergent sequence, complete fuzzy Bi-normed linear space in $P^*(X)$, finite dimensional fuzzy Bi-normed linear space and its properties are discussed. We have introduced fuzzy Bi-normed linear space of triangle inequality theorem, finite dimensional fuzzy subspaces theorem and complete fuzzy Bi-normed linear space theorem are established in fuzzy setting. Since these theorems have many applications is functional analysis, numerical examples are given to clarify the developed theory and proposed Bi-normed linear spaces.

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