**International Journal of Pure and Applied Mathematics**
**Volume 118 No. 6 2018, 381-387**
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
**url:** http://www.ijpam.eu
Special Issue
ijpam.eu

# Total Domination Node Critical and Stable Fuzzy Graphs

R. Jahir Hussain[1] and S. Ameena Banu[2]

[1] *PG and Research Department of Mathematics,*
*Jamal Mohamed College (Autonomous),*
*Tiruchirappalli-620020. India.*
*Email: hssn_jhr@yahoo.com*
[2] *PG and Research Department of Mathematics,*
*Jamal Mohamed College (Autonomous),*
*Tiruchirappalli-620020. India.*
*Email: banuameena3@gmail.com*

**Abstract**

In general $\gamma_f(G)$ can be increased or decreased by removing or adding an arc or node in $G$. In this paper we discussed fuzzy total dominating set, dominating critical node and dominating stable node. Also we examine the effects of fuzzy total domination number when we remove a node from $G$.
**AMS Subject Classification:** 03E72, 05C72
**Key Words:** Fuzzy dominating set, Fuzzy domination number, Fuzzy total dominating set, Fuzzy total domination number, Dominating critical node and Dominating stable node and Fuzzy Total dominating critical node.

## 1 Introduction

Brigham [1] introduced vertex domination critical graphs. Harary et al [2] explained an interesting application in voting situations using the concept of domination. Nagoor Gani [8] and Vijayalakshmi [4] discussed domination critical nodes in fuzzy graph Rosenfeld [9] introduced the notion of fuzzy graph and several fuzzy analogous of graph theoretic concepts such as paths, cycles, connectedness and etc. Somasundaram and Somasundaram [10] discussed domination in fuzzy graphs. Sumner [11] discussed domination critical graphs. Hedetniemi and Cockayne [4] introduced total domination in graphs. Haynes and Henning [5]discussed total domination changing and stable graphs upon vertex removal. (See [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11])

## 2 Preliminaries

A *fuzzy graph* $G = (\sigma, \mu)$ is a non-empty set $V$ together with a pair of functions $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$ such that $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$, where $\sigma(u) \wedge \sigma(v)$ is the minimum of $\sigma(u)$ and $\sigma(v)$. The *underlying crisp graph* of the fuzzy graph $G = (\sigma, \mu)$ is denoted by $G^* = (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V/\sigma(u) > 0\}$ and $\mu^* = \{(u,v) \in V \times V/\mu(u,v) > 0\}$. Let $G = (\sigma, \mu)$ be a fuzzy graph and $\tau$ be any fuzzy subset of $\sigma$, i.e $\tau(u) \leq \sigma(u)$ for all $u$. Then the fuzzy sub graph of $G = (\sigma, \mu)$ induced by $\tau$ is the maximal fuzzy sub graph of $G = (\sigma, \mu)$ that has fuzzy node set $\tau$. Evidently this is just the fuzzy graph $(\tau, \rho)$, where $\rho(u,v) = \tau(u) \wedge \tau(v)$ for all $u, v \in V$.

A fuzzy graph $G = (\sigma, \mu)$ is a *complete fuzzy graph* if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in \sigma^*$. Two nodes $u$ and $v$ are said to be *neighbours* if $\mu(u,v) > 0$. The *strong neighbourhood* of $u$ is $N_s(u) = \{v \in V : (u,v) \text{is a strong arc}\}$. $N_s[u] = N_s(u) \cup \{u\}$ is the *closed strong neighbourhood* of $u$. Let $G = (\sigma, \mu)$ be a fuzzy graph. Two nodes $u$ and $v$ of $G$ are *strong adjacent* if $(u,v)$ is strong arc. The *strong degree* of a node $v$ is the minimum number of nodes that are strong adjacent to $v$. It is denoted by $d_s(v)$. The minimum cardinality of strong neighbourhood $\delta_s(G) = min\{|N_s(u)| : u \in V(G)\}$ and the maximum cardinality of strong neighbourhood $\Delta_s(G) = max\{|N_s(u)| : u \in V(G)\}$. Let $G$ be a fuzzy graph. Let $S$ be a set of vertices in $G$. Let $u \in S$ then the *private neighbourhood* of $u$ is $pn(u,S) = \{v : N_s(u) \cap S = \{u\}\}$. The *external private neighbourhood* $epn(v,S) = pn(u,S) \setminus S$. A node $u$ is called a *fuzzy end node* of $G = (\sigma, \mu)$ if it has atmost one strong neighbour in $G = (\sigma, \mu)$.

A path $\rho$ in a fuzzy graph is a sequence of distinct nodes $u_0, u_1, u_2, ...u_n$ such that $\mu(u_{i-1}, u_i) > 0$; $1 \leq i \leq n$ here $n \geq 0$ is called the *length of the path* $\rho$. The consecutive pairs $(u_{i-1}, u_i)$ are called the *arcs* of the path. A path $\rho$ is called a *cycle* if $u_0 = u_n$ and $n \geq 3$. An arc $(u,v)$ is said to be a *strong arc* if $\mu(u,v) \geq \mu^\infty(u,v)$ and the node $v$ is said to be a *strong neighbour* of $u$. If $\mu(u,v) = 0$ for every $v \in V$ then $u$ is called *isolated node*. Two nodes that are joined by a path are said to be *connected*. The relation connected is a reflexive, symmetric and transitive. The equivalence classes of nodes under this relation are the *connected components* of the given fuzzy graph. A fuzzy graph $G = (\sigma, \mu)$ is *fuzzy bipartite* if it has a spanning fuzzy sub graph $H = (\tau, \pi)$ which is bipartite where for all edges $(u,v)$ not in $H$, weight of $(u,v)$ in $G$ is strictly less than the strength of pair $(u,v)$ in $H$ (i.e) $\mu(u,v) < \pi^\infty(u,v)$. A fuzzy bipartite graph $G$ with fuzzy bipartition $(V_1, V_2)$ is said to be a *complete fuzzy bipartite* if for each node of $V_1$, every node of $V_2$ is a strong neighbour.

Let $G = (\sigma, \mu)$ be a fuzzy graph and $u$ be a node in $G$ then there exist a node $v$ such that $(u,v)$ is a strong arc then we say that $u$ *dominates* $v$. Let $G = (\sigma, \mu)$ be a fuzzy graph. A set $D$ of $V$ is said to be *fuzzy dominating set* of $G$ if every $v \in V - D$ there exist $u \in D$ such that $u$ dominates $v$. A fuzzy dominating set $D$ of $G$ is called a *minimal fuzzy dominating set* of $G$ if no proper subset of $D$ is a fuzzy dominating set. The *fuzzy domination number* $\gamma_f(G)$ of the fuzzy graph $G$ is the smallest number of nodes in any fuzzy dominating set of $G$. A fuzzy dominating set $D$ of a fuzzy graph $G$ such that $|D| = \gamma_f(G)$ is called *minimum fuzzy dominating* *set.*

## 3 Fuzzy Dominating Critical and Stable Nodes

**Definition 1.** Let $G = (\sigma, \mu)$ be a fuzzy graph. A node $v$ of $G$ is said to be *fuzzy dominating critical node* if its removal either increases (or) decreases the fuzzy domination number.

Let $G = (\sigma, \mu)$ be a fuzzy graph. A node $v$ of $G$ is said to be *fuzzy dominating stable node* if its removal does not affect the fuzzy domination number.

We partition the nodes of $G$ into three disjoint sets according to how their removal affects $\gamma_f(G)$. Let $V = V_f^0 \cup V_f^+ \cup V_f^-$ for

$V_f^0 = \{v \in V : \gamma_f(G - v) = \gamma_f(G)\}$

$V_f^+ = \{v \in V : \gamma_f(G - v) > \gamma_f(G)\}$

$V_f^- = \{v \in V : \gamma_f(G - v) < \gamma_f(G)\}$

**Results**

1. If $I$ is the set of all isolated nodes of $G$ then $I \subseteq V_f^-$.

2. If $D$ is a fuzzy dominating set, removing any note in $V - D$ cannot increase the fuzzy domination number therefore removing any node in $D$ increase the $\gamma_f(G)$. In other words $D \subseteq V_f^+$.

## 4 Fuzzy Total Dominating Set and Fuzzy Total Domination Number

**Definition 2.** A set $S$ of nodes in a fuzzy graph $G = (\sigma, \mu)$ is a fuzzy total dominating set if every node of $G$ is adjacent to atleast one node in $S$.

A fuzzy total dominating set $S$ of $G$ is called a *minimal fuzzy total dominating set* of $G$ if no proper subset of $S$ is a fuzzy total dominating set.

The *fuzzy total domination number* $\gamma_{tf}(G)$ of the fuzzy graph $G$ is the smallest number of nodes in any fuzzy total dominating set of $G$.

A fuzzy total dominating set $S$ of a fuzzy graph $G$ such that $|S| = \gamma_{tf}(G)$ is called *minimum fuzzy total dominating set.*

**Observation 1:** For $n \geq 3$, $\gamma_{tf}(P_n) = \gamma_{tf}(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$

**Proposition 1:** If $S$ is a minimal total dominating set of a connected fuzzy graph $G$, then for each node $v \in S$, $|epn(v, s)| \geq 1$.

## 5 Total Domination Node Removal Critical and Stable Fuzzy Graphs

**Definition 3.** A fuzzy graph $G = (\sigma, \mu)$ is *total domination node removal critical* if the removal of an arbitrary node changes the fuzzy total domination number.

A fuzzy graph $G = (\sigma, \mu)$ is *total domination node removal stable* if the removal of an arbitrary node does not changes the fuzzy total domination number.

We partition the nodes of $G$ into three disjoint sets according to how their removal affects $\gamma_{tf}(G)$.
Let $V = V_{tf}^0 \cup V_{tf}^+ \cup V_{tf}^-$ for
$V_{tf}^0 = \{v \in V : \gamma_{tf}(G - v) = \gamma_{tf}(G)\}$
$V_{tf}^+ = \{v \in V : \gamma_{tf}(G - v) > \gamma_{tf}(G)\}$
$V_{tf}^- = \{v \in V : \gamma_{tf}(G - v) < \gamma_{tf}(G)\}$

**Example 1:**

1. Consider a subdivided fuzzy star $G = k_{1,r}^*$ where $r \geq 2$ whose underlying crisp graph is subdivided star. If $v$ is the central node of $G$ then $\gamma_{tf}(G) = r + 1$ while $\gamma_{tf}(G - v) = 2r$. This is an example for removal of node increase the fuzzy total domination number.

2. Let $G = C_n$ where $n \equiv 1, 2(mod 4)$. Then by observation 1 we have $\gamma_{tf}(G-v) < \gamma_{tf}(G)$ for every node $v$ of $G$. This is an example for removal of node decrease the fuzzy total domination number.

3. Let $G = C_n$ where $n \equiv 1, 2(mod 4)$. Then by observation 1 we have $\gamma_{tf}(G-v) = \gamma_{tf}(G)$ for every node $v$ of $G$. This is an example for removal of node does not change the fuzzy total domination number.

**Remarks**

1. A fuzzy graph $G = (\sigma, \mu)$ is $\gamma_{tf}$ critical if the removal of any node from $G$ either increase or decrease the fuzzy total domination number. (ie) $V(G) = V_{tf}^+ \cup V_{tf}^-$.

2. A fuzzy graph $G = (\sigma, \mu)$ is $\gamma_{tf}$ stable if the removal of any node from $G$ does not change the fuzzy total domination number. (ie) $V(G) = V_{tf}^0$.

3. Example for $\gamma_{tf}$ critical is fuzzy path $P_5$ where the fuzzy end nodes of $P_5$ are in $V_{tf}^-$ and the remaining nodes are in $V(G) = V_{tf}^+$.

**Observation 2:** Let $G$ be a fuzzy graph, and let $v \in V_{tf}^-$. For every $\gamma_{tf}(G - v)$ set $S$, $|S| = \gamma_{tf}(G) - 1$ and $S \cap N_s(v) = \phi$.

**Theorem 4.** *Let $G$ be a fuzzy graph without isolated nodes. A node $v$ is in $V_{tf}^-$ if and only if there exists some $\gamma_{tf}(G)$ set $S$ and a node $u \in S$ such that $v \notin S$ and $pn(u, S) = \{v\}$.*

*Proof.* Let $G$ be a fuzzy graph without isolated nodes, and let $v \in V_{tf}^-$. Let $S^*$ be an arbitrary $\gamma_{tf}(G - v)$ set, and let $u \in N_s(v)$. By observation 2, $|S^*| = \gamma_{tf}(G) - 1$ and $S^* \cap N_s(v) = \phi$. Then $S = S^* \cup \{u\}$ is a $\gamma_{tf}(G)$ set such that $v \notin S$ and $pn(u, S) = \{v\}$.
Conversely, assume that there exists a $\gamma_{tf}(G)$ set $S$ such that $v \notin S$ and $pn(u, S) = \{v\}$ for some $u \in S$. The set $S \backslash \{u\}$ is a total dominating set for $G - v$ of cardinality $\gamma_{tf}(G) - 1$. Hence $v \in V_{tf}^-$. □

**Theorem 5.** *Let $G$ be a fuzzy graph and $v \in V$. Then $v \in V_{tf}^{+}$ if and only if $v \in A_t(G)$[where $A_t(G)$ is the set of nodes of $G$ which are contained in every $\gamma_{tf}(G)$set] and either $v \in S(G)$[where $S(G)$ is the set of nodes which are adjacent to fuzzy end nodes] or no subset $S$ of $V \backslash N_s[v]$ with cardinality $\gamma_{tf}(G)$ is a total dominating set for $G - v$.*

*Proof.* Let $G$ be a fuzzy graph, and let $v \in V_{tf}^{+}$. If there exists a $\gamma_{tf}(G)$ set $D$ such that $v \notin D$, then the set $D$ is a total dominating set for $G - v$. Hence $\gamma_{tf}(G - v) \leq |D| = \gamma_{tf}(G)$, and so $v \notin V_{tf}^{+}$, a contradiction. Therefore $v \in A_t(G)$.

If $v \in S(G)$, we are finished. Hence we may assume that $v \notin S(G)$. If there exists some set $S \subseteq V \backslash N_s[v]$ of cardinality $\gamma_{tf}(G)$ which total dominates $G-v$, then $\gamma_{tf}(G - v) \leq |S| = \gamma_{tf}(G)$, and so $v \notin V_{tf}^{+}$, a contradiction. Therefore no subset $S$ of $V \backslash N_s[v]$ with cardinality $\gamma_{tf}(G)$ is a total dominating set for $G - v$. This proves the necessity.

For the sufficiency, suppose that $v \in A_t(G)$ and either $v \in S(G)$ or no subset $S$ of $V \backslash N_s[v]$ with cardinality $\gamma_{tf}(G)$ is a total dominating set for $G - v$. If $v \in S(G)$, then $\gamma_{tf}(G - v) = \infty$ and so $v \in V_{tf}^{+}$ as desired. Hence we may assume that $v \notin S(G)$. We now consider an arbitrary $\gamma_{tf}(G - v)$ set $S$.

On one hand, if $S \subseteq V \backslash N_s[v]$, then by assumption, $|S| > \gamma_{tf}(G)$. On the other hand, if $S \nsubseteq V \backslash N_s[v]$, then $S$ contains a node in $N_s(v)$. But then $S$ is a total dominating set of $G$. However since $v \notin S$ and $v \in A_t(G)$, the set $S$ is not a minimum total dominating set of $G$, implying that $|S| > \gamma_{tf}(G)$. In both cases we have $\gamma_{tf}(G - v) = |S| > \gamma_{tf}(G)$, and so $v \in V_{tf}^{+}$ as desired. □

**Corollary 6.** *Let $G$ be a fuzzy graph with $n$ nodes. Then the following holds.*

1. *$|V_{tf}^{+}| \leq \gamma_{tf}(G)$.*
2. *If $G$ is a $\gamma_{tf}$ critical, then $|V_{tf}^{-}| \geq n - \gamma_{tf}(G)$.*
3. *If $G$ is a $\gamma_{tf}$ critical and $V = V_{tf}^{+}$ then $G = n/2K_2$.*

*Proof.* Part (1) follows from theorem 7 and the observation that $\gamma_{tf}(G) \geq |A_t(G)|$. Part (2) follows from part (1) and the observation that $V(G) = V_{tf}^{+} \cup V_{tf}^{-}$ if $G$ is a $\gamma_{tf}$ critical graph. If $G$ is a $\gamma_{tf}$ critical graph with $V(G) = V_{tf}^{+}$, then by theorem 7, $A_t(G) = V$, implying that $\gamma_{tf}(G) = |V|$ and $G = kK_2$ for some $k \geq 1$, there by establishing part (3). □

**Lemma 1:** If $uv$ is an arc in a fuzzy graph $G$, where $u \in V_{tf}^{+}$ and $v \in V_{tf}^{-}$, then $v$ is a fuzzy end node of $G$.

*Proof.* Assume that $uv$ is an arc in $G$, where $u \in V_{tf}^{+}$ and $v \in V_{tf}^{-}$, then $v$ is not fuzzy end node of $G$. Thus $d_s(v) \geq 2$. Let $S$ be any $\gamma_{tf}(G - v)$ set. By observation 2, $|S| = \gamma_{tf}(G) - 1$ and $S \cap N_s(v) = \phi$.

In particular $u \notin S$. Let $w$ be a strong neighbour of $v$ different from $u$. Then the set $S \cup \{w\}$ is a $\gamma_{tf}(G)$ set not containing $u$, and so $u \notin A_t(G)$. Hence by theorem 7, $u \notin V_{tf}^{+}$, a contradiction. Therefore $v$ is a fuzzy end node of $G$. □

**Corollary 7.** *If $G$ is a connected $\gamma_{tf}$ critical fuzzy graph with $\delta_s(G) \geq 2$, then $V = V_{tf}^{+}$ or $V = V_{tf}^{-}$.*

**Corollary 8.** *A fuzzy graph G is a* $\gamma_{tf}$ *critical with* $V_{tf}^{+} = \phi$ *if and only if* $\delta_s(G) \geq 2$*, and for every* $v \in V$*, there exists some* $\gamma_{tf}(G)$ *set S and some node* $u \in S$ *such that* $v \notin S$ *and* $pn(u, S) = \{v\}$*.*

**Corollary 9.** *A fuzzy graph G is a* $\gamma_{tf}$ *stable if and only if* $\delta_s(G) \geq 2$ *and for every* $v \in V$*, both of the following holds.*

1. *There is no* $\gamma_{tf}(G)$ *set S such that* $v \notin S$ *and* $pn(u, S) = \{v\}$ *for some node* $u \in S$*.*
2. *Either* $v \notin A_t(G)$ *or* $v \in A_t(G)$ *and there exist a total dominating set S in* $G - v$ *such that* $|S| = \gamma_{tf}(G)$ *and* $S \subseteq V \backslash N_s[v]$*.*

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