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Criss-cross Method for Solving the Fuzzy Linear Complementarity Problem

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Abstract

In this paper the criss-cross method is proposed to solve the Fuzzy Linear Complementarity Problems (FLCP). Here, some basic properties of sufficient fuzzy matrices and necessary and sufficient properties for the finiteness of the criss-cross method are discussed.

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Key Words and Phrases: Fuzzy linear complementarity problem, sufficient fuzzy matrices, criss-cross method.

1 Introduction

The LCP is one of the most widely studied problems of mathematical programming. Several methods have been developed for solving LCPs(see eg. Cottle [3], Cottle and Dantzig [5], Lemke [16]). These methods utilize different pivot rules. There also exist several nonpivot methods. An excellent survey of the existing methods and the classification of matrices for LCPs can be found in Murty's [17] book. Nowadays the LCP is a subject of research on different approches: Polynomial methods,combinatorial abstraction, Identification of matrix classes

This paper examines sufficient FLCPs and their solution by the criss-cross method. The criss-cross method is a simple, purely combinatorial method, so the object of this paper is to characterize a fuzzy matrix class by the finiteness of a combinatorial method. The criss-cross method as though to have been discovered by Terlaky [18,19] and later independently by Wang [20]. In minimal index type methods there is no minimal ratio test. This cuts down the computational effort per iteration.

The criss-cross method is known to be finite for LCPs with positive semidefinite bisymmetric matrices and with P-matrices. It is also a simple finite algorithm for oriented matroid programming problems. The properties that are necessary to guarantee the applicability and finiteness of the criss-cross method are studied in this paper. We will show that the criss-cross method is finite for sufficient FLCPs. Further, it is also proved that a matrix M is sufficient if and only if the criss-cross method processes problems $(\tilde{q}, \tilde{M})$ and $(\tilde{q}, \tilde{M}^T)$ with any parameter vector $\tilde{q}$. As for terminology, we say that the criss-cross method processes a problem if it finds a solution or detects infeasibility in a finite number of steps.

The paper is organized as follows: Section 2 provides the preliminaries and the basic properties of (column, row) sufficient fuzzy matrices. The criss-cross method is stated in section 3, and the properties that are necessary to execute it and guarantee its finiteness are presented in section 4. The characterization of the class of sufficient fuzzy matrices by the criss-cross methd is discussed in section 5.

2 Preliminaries

Definition 1. Given the $n \times n$ matrix M and the n-dimensional vector q, the Linear Complementarity Problem (LCP) consists in finding non - negative vectors w and z which satisfy,

$$w - Mz = q \tag{1}$$
$$w_i, z_i \geq 0, \text{ for } i = 1, 2, ..., n. \tag{2}$$
$$w_i z_i = 0, \text{ for } i = 1, 2, ..., n. \tag{3}$$

Given the non-negativity of the vectors w and z, (3) requires that $w_i z_i = 0$ for $i = 1,\ 2,\ .....,\ n$. Two such vectors are said to be complementarity. A solution (w, z) to the LCP is called a complementarity feasible solution, if it is a basic feasible solution to (1) and (2) with one of the pairs (w_i, z_i) is basic.

Definition 2. Assume that all parameters in (1)-(3) are fuzzy and are described by fuzzy numbers. Then the following fuzzy linear complementarity problem can be obtained by replacing crisp parameters with fuzzy number.

$$\tilde{w} - \tilde{M}\tilde{z} = \tilde{q} \tag{4}$$
$$\tilde{w_i}, \tilde{z_i} \geq 0, \text{ } for \text{ } i = 1, 2, \ldots \ldots, n \text{ and} \tag{5}$$
$$\tilde{w_i}\tilde{z_i} = 0, \text{ for } i = 1, 2, \ldots \ldots, n. \tag{6}$$

The pair $(\tilde{w_i}, \tilde{z_i})$ is said to be a pair of fuzzy complementarity variables.

Definition 3. A fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(x, \mu_A(x)) : x\epsilon A,\ \mu_A(x)\epsilon[0, 1]\}$. In the pair $(x, \mu_A(x))$, the first element x belong to the classical set A, the second element $\mu_A(x)$, belong to the interval $[0, 1]$, called membership function.

Definition 4. A fuzzy matrix $\tilde{M}$ is Called

(i) row sufficient if $\tilde{X}\tilde{M}^T\tilde{x} \leq \tilde{0}$ implies $\tilde{X}\tilde{M}^T\tilde{x} = \tilde{0}$ for every vector $\tilde{x}$;

(ii) column sufficient if $\tilde{X}\tilde{M}\tilde{x} \leq \tilde{0}$ implies $\tilde{X}\tilde{M}\tilde{x} = \tilde{0}$ for every vector $\tilde{x}$;

(iii) sufficient if it is both row and column sufficient.

Basic properties of sufficient fuzzy matrices:

Proposition 5. *Every principal rearrangement of a (column, row) fuzzy sufficient matrix is (column, row) fuzzy sufficient.*

Proposition 6. *Let $\tilde{D}$ be an invertible diagonal fuzzy matrix. Then a fuzzy matrix $\tilde{M}$ is (column, row) sufficient if and only if $\tilde{D}\tilde{M}\tilde{D}$ is (column, row) sufficient.*

Proposition 7. *Every principal fuzzy submatrix of a (column, row) sufficient fuzzy matrix is (column, row) sufficient.*

Proposition 8. *Both column and row sufficient fuzzy matrices have nonnegative principal fuzzy submatrices, and hence nonnegative diagonal elements.*

Proposition 9.
1. *Let $\tilde{M}$ be row sufficient with $\tilde{m}_{ii} = \tilde{0}$ for some i. If $\tilde{m}_{ij} \neq \tilde{0}$ for some j, then $\tilde{m}_{ji} \neq \tilde{0}$, and in this case $\tilde{m}_{ij}\tilde{m}_{ji} < \tilde{0}$.*
2. *Let $\tilde{M}$ be column sufficient with $\tilde{m}_{ii} = \tilde{0}$ for some i. If $\tilde{m}_{ji} \neq \tilde{0}$ for some j, then $\tilde{m}_{ij} \neq \tilde{0}$, and in this case $\tilde{m}_{ji}\tilde{m}_{ij} < \tilde{0}$.*
3. *Let $\tilde{M}$ be sufficient with $\tilde{m}_{ii} = \tilde{0}$ for some i. One has $\tilde{m}_{ij} \neq \tilde{0}$ for some j, if and only if $\tilde{m}_{ji} \neq \tilde{0}$, and then $\tilde{m}_{ij}\tilde{m}_{ji} < \tilde{0}$.*

Let a diagonal element $\tilde{m}_{ii}$ be zero for some i. Then as a consequence of Proposition 9 for(row, column) sufficient fuzzy matrices have:

1. For row sufficient fuzzy matrices: If $\tilde{m}_{ji} \geq \tilde{0}$ for all j, then $\tilde{m}_{ij} \leq \tilde{0}$ for all j. If $\tilde{m}_{ji} \leq \tilde{0}$ for all j, then $\tilde{m}_{ij} \geq \tilde{0}$ for all j.
2. For column sufficient fuzzy matrices: If $\tilde{m}_{ij} \geq \tilde{0}$ for all j, then $\tilde{m}_{ji} \leq \tilde{0}$ for all j. If $\tilde{m}_{ij} \leq \tilde{0}$ for all j, then $\tilde{m}_{ji} \geq \tilde{0}$ for all j.
3. For sufficient fuzzy matrices: If $\tilde{m}_{ij} \leq \tilde{0}$ for all j, if and only if $\tilde{m}_{ji} \geq \tilde{0}$ for all j. Moreover, $\tilde{m}_{ij} \geq \tilde{0}$ for all j, then $\tilde{m}_{ji} \leq \tilde{0}$ for all j.

Proposition 10. *Any principal pivotal fuzzy transform of a (column, row) sufficient fuzzy matrix is (column, row) sufficient.*

Proposition 11. *A 2×2 fuzzy matrix $\tilde{M}$ is sufficient if and only if for every principal pivotal fuzzy transform $\overline{\tilde{M}}$ of $\tilde{M}$.*

1. *$\overline{\tilde{m}_{ii}} \geq \tilde{0}$ and*
2. *for $i = 1,\ 2$, if $\overline{\tilde{m}_{ii}} = \tilde{0}$, then either $\overline{\tilde{m}_{ij}} = \overline{\tilde{m}_{ij}} = \tilde{0}$ or $\overline{\tilde{m}_{ij}}\ \overline{\tilde{m}_{ij}} < \tilde{0}$ for $j \neq i$.*

Proposition 12. *A fuzzy matrix $\tilde{M}$ is sufficient if and only it every principal pivotal fuzzy transform $\overline{\tilde{M}}$ of $\tilde{M}$ is sufficient of order 2 (i. e., every 2×2 principal fuzzy submatrix of $\overline{\tilde{M}}$ is sufficient).*

Lemma 13. *For any two bases $\tilde{B}$ and $\tilde{B}'$ we have that $\tilde{t}_i$ is orthogonal to $\tilde{t}_k^{'}$ for all $i \in \tilde{J}_B$ and $k \in \overline{\tilde{J}_{B'}}$, where $\tilde{t}_i$ is defined by the basis $\tilde{B}$ and $\tilde{t}_k^{'}$ is defined by the basis $\tilde{B}'$.*

Proof. The orthogonality of the two vectors is obvious if $\tilde{B} = \widetilde{B}'$. This implies that the subspace spanned by $\{\underline{\tilde{t}_i} : i \in \tilde{J}_B\}$ is the orthogonal complement of the subspace spanned by $\{\tilde{t}_k : k \in \overline{\tilde{J}}_B\}$. Since pivoting (changing the basis) preserves the row space of canonical tableaus (the first subspace above), the orthogonal complement remains also the same. This implies the lemma. □

3 The Criss-Cross method for fuzzy linear complementarity problems

Let an FLCP be given as it is presented in section 2. The initial basis is given by the matrix $\tilde{E}$, and the initial tableau is [-$\tilde{M}$, $\tilde{E}$, $\tilde{q}$]. A tableau is called complementary if the corresponding solution satisfies the complementarity condition. The above-defined initial tableau is complementary. For simplicity the nonbasic part of any complementary canonical tableau will also be denoted by $-\widetilde{M}$ if no confusion is possible. Note that the nonbasic part of any complementary canonical tableau is a principal pivotal fuzzy transform of the original matrix $-\widetilde{M}$. This algorithm STOPs if the problem is processed, while EXIT is used if it fails to process the problem. The Criss-Cross method is defined as follows.

Criss-Cross method:

Initialization:

Let the starting basis be defined by $\tilde{w}$, and let $\tilde{w}= \tilde{q}$, $\tilde{z} = \tilde{0}$ be the initial solution.

The initial tableau is given by [-$\tilde{M}$, $\tilde{E}$, $\tilde{q}$].

Pivot rule:

We have a complementary basis and the corresponding tableau.

Leaving variable selection:

Let $k := min\{i : \tilde{w}_i < \tilde{0}$ or $\tilde{z}_i < \tilde{0}\}$.

If there is no such k, then STOP; a feasible complementary solution has been found. (Without loss of generality we may assume that $\tilde{w}_k < \tilde{0}$).

Entering variable selection:

Diagonal pivot:

If $-\tilde{m}_{kk} < \tilde{0}$, then make a diagonal pivot and repeat the procedure. (Here $\tilde{w}_k$ leaves and $\tilde{z}_k$ enters the basis).

If $-\tilde{m}_{kk} > \tilde{0}$, then EXIT.

If $-\tilde{m}_{kk} = \tilde{0}$, select an exchange pivot.

Exchange pivot:

We know that $\tilde{m}_{kk} = \tilde{0}$ in this case. Let $r := min\{j : -\ \tilde{m}_{kk} < \tilde{0}$ or $-\tilde{m}_{jk} > \tilde{0}\}$.

If there is no r, then STOP; FLCP is infeasible.

If there is an r and $\tilde{m}_{rk}\tilde{m}_{kr} \geq \tilde{0}$, then EXIT.

If there is an r and $\tilde{m}_{rk}\tilde{m}_{kr} < \tilde{0}$, then make an exchange pivot on (r, k) and repeat the procedure. (Here $\tilde{w}_k$ and $\tilde{z}_r$ leave and $\tilde{z}_k$ and $\tilde{w}_r$ enter the basis).

4 Sufficient and necessary properties for the finiteness of the Criss-Cross method

Let $\widetilde{\mathcal{M}}$ be the class of fuzzy matrices such that for each $\tilde{M} \in \widetilde{\mathcal{M}}$ and for each vector $\tilde{q} \in R_n$ the problem ($\tilde{q}$, $\tilde{M}$) is processed successfully by the Criss-Cross method.

Property 1: If $\tilde{M} \in \widetilde{\mathcal{M}}$, the diagonal elements of any principal pivotal fuzzy transform of $-\tilde{M}$ are nonpositive.

Property 2: If $-\tilde{M}_{kk} = \tilde{0}$ for some k, then $-\tilde{m}_{kj} < \tilde{0}$ if and only if $-\tilde{M}_{jk} > \tilde{0}$ for any j.

Property 3: For a problem $(\tilde{q},\ \tilde{M})$ the pairs of cases AB, CD, AC and BD are exclusive for any index 1≤k≤n;

A: We have a complementary tableau with $\tilde{w}_i \geq \tilde{0}$, $\tilde{z}_i \geq \tilde{0}$ for i<k, and $\tilde{w}_k$=$\tilde{0}$, $\tilde{z}_k$<$\tilde{0}$.

B: We have a complementary tableau with $\tilde{w}_i \geq \tilde{0}$, $\tilde{z}_i \geq \tilde{0}$ for i<k, and $\tilde{w}_k < \tilde{0}$, $\tilde{z}_k = \tilde{0}$.

C: We have a complementary tableau with $\tilde{z}_s$< $\tilde{0}$ for some s<k, $\tilde{m}_{si} \geq \tilde{0}$ for i<k, $\tilde{m}_{ss}$= $\tilde{0}$ and $\tilde{m}_{sk}$< $\tilde{0}$; and symmetrically $\tilde{m}_{is} \leq \tilde{0}$ for i<k and $\tilde{m}_{ks}$>$\tilde{0}$.

D: We have a complementary tableau with $\tilde{w}_s$<$\tilde{0}$ for some s<k, $\tilde{m}_{si} \geq \tilde{0}$ for i<k, $\tilde{m}_{ss}$= $\tilde{0}$ and $\tilde{m}_{sk}$< $\tilde{0}$; and symmetrically $\tilde{m}_{is} \leq \tilde{0}$ for i<k and $\tilde{m}_{ks}$>$\tilde{0}$.

Let $\widetilde{\mathcal{M}}'$ denote the class of fuzzy matrices for which properties 1, 2 and 3 hold for any vector $\tilde{q}$ and which is complete with respect to these properties. Observe that if a matrix belongs to $\widetilde{\mathcal{M}}'$, then its transpose not necessarily is in this class.

Theorem 14. $\widetilde{\mathcal{M}}' \subseteq \widetilde{\mathcal{M}}$.

5 The Criss-Cross method and sufficient fuzzy linear complementarity problems

Let us consider the class of sufficient fuzzy matrices, denoted by $\widetilde{\mathcal{F}_u}$. Note that $\widetilde{\mathcal{F}_u}$ is a closed complete class.

Theorem 15. $\widetilde{\mathcal{F}_u} = \widetilde{\mathcal{M}_s}'$

Proof. First prove that $\widetilde{\mathcal{F}_u} \subset \widetilde{\mathcal{M}_s}'$. A fuzzy matrix is sufficient if and only if its transpose is sufficient, so it is enough to prove that the three properties hold for sufficient fuzzy matrices.

Every principal fuzzy transform (see proposition 10) and any principal fuzzy submatrix (see proposition 7) of a sufficient fuzzy matrix is sufficient, so if the required properties hold, then they hold for principal fuzzy submatrices and principal pivotal fuzzy transforms as well.

Property 1 follows from proposition 8. Property 2 follows from proposition 9. To prove property 3 we only have to prove that cases A and B are exclusive. The others follow immediately from the orthogonality property.

Now let us assume to the contrary that for a sufficient FLCP both cases A and B occur. Let the actual complementary solutions be denoted by $(\tilde{z},\ \tilde{w})$ and $(\tilde{z}',\ \tilde{w}')$ respectively. Without loss of generality we may assume that $\tilde{z}_n < \tilde{0}$, $\tilde{w}_n = \tilde{0}$, $\tilde{z}_n' = \tilde{0}$, $\tilde{w}_n' < \tilde{0}$, and all the other coordinates are nonnegative. Then using (1) and the sign and complementarity properties of vectors $(\tilde{z},\ \tilde{w})$ and $(\tilde{z}',\ \tilde{w}')$, we have $(\tilde{Z}\text{-}\tilde{Z}')\ \tilde{M}(\tilde{z}\text{-}\tilde{z}') = (\tilde{Z}\text{-}\tilde{Z}')(\tilde{w}\text{-}\tilde{w}') \leq \tilde{0}$.

For the nth coordinate we have $(\tilde{z}_n\text{-}\tilde{z}_n')(\tilde{w}_n\text{-}\tilde{w}_n') = -\tilde{z}_n\tilde{w}_n' < \tilde{0}$, which contradicts the (column) sufficiency of fuzzy matrix $\tilde{M}$.

On the other hand, if $\tilde{M} \in \widetilde{\mathcal{M}_s}^{\prime}$, then properties 1 and 2 holds for $\tilde{M}$ and $\tilde{M}^T$, and hence propositions 11 and 12 imply that fuzzy matrix $\tilde{M}$ is sufficient. So the equivalence of $\widetilde{\mathcal{M}_s}^{\prime}$ and the class of sufficient fuzzy matrices is proved. □

Corollary 16. *Let $(\tilde{q},\ \tilde{M})$ be a given FLCP, where $\tilde{q}$ is an arbitrary vector and $\tilde{M}$ is a sufficient fuzzy matrix. Then the criss-cross method will process $(\tilde{q},\ \tilde{M})$ in a finite number of steps.*

Theorem 17. $\widetilde{\mathcal{F}_u} = \tilde{M}_s^{\prime} = \widetilde{\mathcal{M}_s}$.

Proof. Theorems 14 and 15 state that $\widetilde{\mathcal{F}_u} = \widetilde{\mathcal{M}_s}^{\prime} \subset \widetilde{\mathcal{M}_s}$. So one only has to prove that if a fuzzy matrix $\tilde{M}$ is not sufficient, then it does not belong to $\widetilde{\mathcal{M}_s}$. If $\tilde{M}$ is not sufficient, then propositions 11 and 12 imply that either property 1 or property 2 does not hold, and so with a properly chosen vector $\tilde{q}$ the criss-cross method cannot process the problem. □

6 Conclusion:

In this paper, criss-cross method for solving the fuzzy linear complementarity problem is suggested. Some basic properties of sufficient fuzzy matrices are discussed and also the finiteness of the criss-cross method is discussed.

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