Accurate 2-Domination in Fuzzy Graphs using Strong Arcs

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Abstract

In this paper, the concept of accurate 2-dominating set and accurate 2-domination number of a fuzzy graph was introduced. In a fuzzy graph G, a subset D of V is said to be an accurate 2-dominating set of G, if V − D has no 2-dominating set of same fuzzy cardinality |D|. The minimum cardinality taken over all accurate 2-dominating sets of G is called the accurate 2-domination number of G and it is denoted as γ(2 fark)(G). Here, some properties of accurate 2-dominating sets and also some bounds of accurate 2-domination numbers are found.

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1 Introduction

Ore [9] and Berge [1] initiated the study on dominating sets in graphs. Then, the domination number and the independent domination number of graphs are introduced by Cockayne and Hedetniemi [2]. The n-domination in graphs was introduced by Fink and Jacobson [3] in the year 1985. Accurate domination in graphs was introduced by Kulli and Kattimani [4].

(See [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12])

2 Preliminaries

**Definition 1.** A fuzzy graph \( G = (\sigma, \mu) \) is a pair of functions \( \sigma : V \rightarrow [0, 1] \) and \( \mu : V \times V \rightarrow [0, 1] \), where for all \( x, y \in V \) we have \( \mu(x, y) \leq \sigma(x) \wedge \sigma(y) \).

**Definition 2.** A fuzzy graph \( H = (\tau, \rho) \) is called a fuzzy subgraph of \( G \) if \( \tau(v_i) \leq \sigma(v_i) \) for all \( v_i \in V \) and \( \rho(v_i, v_j) \leq \mu(v_i, v_j) \) for all \( v_i, v_j \in V \).

**Definition 3.** An arc \((u, v)\) in a fuzzy graph \( G \) is said to be strong, if \( \mu^\infty(u, v) = \mu(u, v) \) and then the nodes \( u \) and \( v \) are said to be strong neighbours. The strong neighbourhood of the node \( u \) is defined as \( N_S(u) = \{v \in V : (u, v) \text{ is a strong arc}\} \).

**Definition 4.** The strong neighbourhood degree of a node \( u \) is defined as \( d_{N_S}(u) = \sum_{v \in N_S(u)} \sigma(v) \). The minimum strong neighbourhood degree of a fuzzy graph \( G \) is defined as \( \delta_{N_S}(G) = \min \{d_{N_S}(u) / u \in V \} \). The maximum strong neighbourhood degree of a fuzzy graph \( G \) is defined as \( \Delta_{N_S}(G) = \max \{d_{N_S}(u) / u \in V \} \).

**Definition 5.** The underlying crisp graph of a fuzzy graph \( G = (\sigma, \mu) \) is denoted by \( G^* = (\sigma^*, \mu^*) \), where \( \sigma^* = \{v_i \in V / \sigma(v_i) > 0\} \) and \( \mu^* = \{(v_i, v_j) \in V / \mu(v_i, v_j) \text{ is a strong arc}\} \).

**Definition 6.** In a fuzzy graph \( G \). A subset \( D \) of \( V \) is said to be dominating set of \( G \), if every node in \( V - D \) has atleast one strong neighbour in \( D \). The minimum cardinality taken over all dominating sets of fuzzy graph \( G \) is called the domination number of \( G \) and it is denoted by \( \gamma_f(G) \).

**Definition 7.** In a fuzzy graph \( G \). A subset \( D \) of \( V \) is said to be 2- dominating set of \( G \), if every node in \( V - D \) has at least two strong neighbours in \( D \). The minimum cardinality taken over all 2- dominating sets of fuzzy graph \( G \) is called the 2- domination number of \( G \) and it is denoted by \( \gamma_{f2}(G) \).

**Definition 8.** In a fuzzy graph \( G \). A subset \( D \) of \( V \) is said to be an accurate dominating set of \( G \), if \( V - D \) has no dominating set of same fuzzy cardinality \( |D| \). The minimum cardinality taken over all accurate dominating sets of fuzzy graph \( G \) is called the accurate domination number of \( G \) and it is denoted by \( \gamma_{fa}(G) \).
3 Accurate 2-domination in fuzzy graphs

In this section, we define accurate 2-dominating set and accurate 2-domination number of a fuzzy graph with suitable example. We also discuss some properties on accurate 2-domination number of fuzzy graphs using strong arcs.

**Definition 9.** A 2-dominating set $D$, where $D \subseteq V$, of fuzzy graph $G$ is said to be an accurate 2-dominating set of $G$ if $V - D$ has no 2-dominating set of same fuzzy cardinality $|D|$. The minimum cardinality taken over all accurate 2-dominating sets of $G$ is called the accurate 2-domination number of $G$ and it is denoted by $\gamma_{fa2}(G)$.

**Example 10.** Here $D_1 = \{a, d, f\}$, $D_2 = \{b, c, e\}$, $D_3 = \{e, f\}$, $D_4 = \{a, c\}$, $D_5 = \{a, b\}$, $D_6 = \{a, b, c, d\}$, $D_7 = \{a, b, e, f\}$, $D_8 = \{c, d, e, f\}$, $D_9 = \{d, f\}$, $D_{10} = \{c, e\}$ are some of the dominating set of fuzzy graph $G$. $D_1$, $D_2$, $D_6$, $D_7$, $D_8$ are the accurate 2-dominating sets.

The fuzzy cardinality of $|D_1|$, $|D_2|$, $|D_6|$, $|D_7|$ and $|D_8|$ are 2.3, 1.8, 2.6, 2.7 and 2.9 respectively.

Then, the accurate 2-domination number is

$\gamma_{fa2}(G) = \min\{|D_1|, |D_2|, |D_6|, |D_7|, |D_8|\}$

$\gamma_{fa2}(G) = \min\{2.3, 1.8, 2.6, 2.7, 2.9\}$

$\gamma_{fa2}(G) = 1.8$

Therefore, the accurate 2-domination number is 1.8 and the minimum accurate 2-dominating set is $D_2$.

**Theorem 11.** In a fuzzy graph $G$, if there exists a node $v \in V$, whose strong neighbours $|N_S(v)| = 0$ or 1, then $v$ belongs to every accurate 2-dominating set of $G$.

**Proof.** Let $G$ be a fuzzy graph.

Let $v \in V(G)$, be a node which has strong neighbour atmost one. i.e., $|N_S(v)| = 0$ or 1.

**Case (i):** Assume that, $v$ doesnot have any strong neighbours in $G$. i.e., $|N_S(v)| = 0$.

If $v \in V - D$, then either any node in accurate 2-dominating set $D$ or any nodes in $V(G)$ cannot dominate $v$. So, $v$ must be dominated by itself, i.e., $v \in D$.

Hence, if $|N_S(v)| = 0$, then $v$ belongs to every accurate 2-dominating set of $G$.

**Case (ii):** Assume that, $v$ has only one strong neighbour in $G$, $|N_S(v)| = 1$.
Let \( u \) be the strong neighbours of \( v \) in \( G \).

Suppose, \( u \) dominates \( v \), then \( u \in D \) and \( v \in V - D \).

Since, \( v \) has only one strong neighbours in \( D \) then \( D \) cannot be an accurate 2-dominating set of \( G \). Therefore, \( v \) should be dominated by itself.

Hence, \( |N_S(v)| = 1 \), then \( v \) belongs to every accurate 2-dominating set of \( G \).

**Corollary 12.** For any fuzzy graph \( G \) if \( |N_S(v)| = 0 \) or \( 1 \), then the accurate 2-dominating number and 2-domination number of \( G \) must be same, i.e., \( \gamma_{fa2}(G) = \gamma_{f2}(G) \).

**Proof.** Let \( G \) be any fuzzy graph and a subset \( D \subseteq V \) be a 2-dominating set of \( G \), then every nodes in \( V - D \) must have two strong neighbours in \( D \).

By theorem 11, if \( |N_S(v)| = 0 \) or \( 1 \), then \( v \) belongs to every accurate 2-dominating set of \( G \). So, \( v \) must belongs to \( D \).

Since, \( v \in D \) and \( v \) has atmost one strong neighbours in \( V(G) \), then \( V - D \) does not have any 2-dominating set of same fuzzy cardinality \( |D| \). Therefore, the 2-dominating set \( D \) itself forms an accurate 2-dominating set of \( G \).

Hence the proof.

**Theorem 13.** Every accurate 2-dominating set of a fuzzy graph \( G \) is also a 2-dominating set of \( G \).

**Proof.** Let \( G \) be a fuzzy graph and \( D \) a subset \( V \), be an accurate 2-dominating set of \( G \). Since, \( D \) is an accurate 2-dominating set, then every nodes in \( V - D \) has atleast two strong neighbours in \( D \).

That is, each node in \( V - D \) is dominated by atleast two nodes in \( D \).

Therefore, every accurate 2-dominating set \( D \) itself forms 2- dominating set of fuzzy graph \( G \).

**Theorem 14.** Every accurate 2-dominating set of a fuzzy graph \( G \) is also an accurate dominating set of \( G \).

**Proof.** Let \( G \) be a fuzzy graph and a subset \( D \subseteq V \), be an accurate 2-dominating set of \( G \).

Since, \( D \) is an accurate 2-dominating set, then every nodes in \( V - D \) has atleast two strong neighbours in \( D \), i.e., every nodes in \( V - D \) will be dominated by atleast two nodes in \( D \). Therefore, \( D \) is also a dominating set of \( G \) and if \( V - D \) does not contains any dominating set of same fuzzy cardinality \( |D| \), Then \( D \) itself forms an accurate dominations set of \( G \).

**Theorem 15.** For any fuzzy graph \( G \), \( \gamma_{f2}(G) \leq \gamma_{fa2}(G) \).

**Proof.** Assume that, \( D \subseteq V \) be the minimum accurate 2-dominating set of fuzzy graph \( G \).

By theorem 13, every accurate 2-dominating set of a fuzzy graph \( G \) is also a 2-dominating set of \( G \). Then, every nodes in \( V - D \) has atleast two strong neighbours in \( D \), therefore, every node \( u \in V - D \) is dominated by atleast two nodes in \( D \). Therefore, \( D \) is a 2-dominating set of \( G \). Hence, \( \gamma_{f2}(G) \leq \gamma_{fa2}(G) \).
Theorem 16. For any fuzzy graph $G$, $\gamma_{fa}(G) \leq \gamma_{fa2}(G)$.

Proof. Assume that, $D \subseteq V$ be the minimum accurate 2-dominating set of fuzzy graph $G$.

By theorem 14, every accurate 2-dominating set of a fuzzy graph $G$ is also an accurate dominating set of $G$. Then, every nodes in $V - D$ has at least two strong neighbours in $D$, therefore, every node $u \in V - D$ is dominated by at least two nodes in $D$. Therefore, $D$ is also a dominating set of $G$ and if $V - D$ does not have any dominating set of same fuzzy cardinality $|D|$. Then, $D$ itself forms an accurate dominating set of $G$. Hence, $\gamma_{fa}(G) \leq \gamma_{fa2}(G)$.

Theorem 17. For any fuzzy graph $G$. If $D$ be an accurate 2-dominating set of fuzzy graph $G$, then $V - D$ need not to be an accurate 2-dominating set of $G$.

Proof. Let $G$ be a fuzzy graph and $D$, a subset of $V$, be an accurate 2-dominating set of $G$.

Case (i): By theorem 11, if $|N_S(v)| = 0$ or 1, then $v$ belongs to every accurate 2-dominating sets of $G$. That is, $v \in D$, has at most one strong neighbour in $V - D$. Therefore, $V - D$ has at most one strong neighbour which dominates $v$. Hence, $V - D$ cannot be a 2-dominating set of $G$ and also it cannot be an accurate 2-dominating set of $G$.

Case (ii): Assume that, $D$ be an accurate 2-dominating set of $G$. If every node $v \in D$ has at least two strong neighbours in $V - D$ and the fuzzy cardinality of $|V - D| \neq |D|$, then $V - D$ is an accurate 2-dominating set of $G$.

Therefore by case(i) and (ii), if $D$ be any accurate 2-dominating set of $G$, then $V - D$ need not be an accurate 2-dominating set of fuzzy graph $G$.

Theorem 18. For any fuzzy graph $G$, $\gamma_{fa2}(G) \leq |V| - \gamma_{fa}(G) + 1$.

Proof. Assume that, a subset $D$ of $V$, be the minimum dominating set of fuzzy graph $G$. Therefore, for any node $v \in D$, $(V - D) \cup \{v\}$ forms an accurate dominating set of $G$. And let us denote it as $A = (V - D) \cup \{v\}$.

Hence, if $V - A$ has at least two strong neighbours in $A$, then the accurate dominating set $A$ itself forms an accurate 2-dominating set of $G$.

That is, $|A| \leq |(V - D) \cup \{v\}|$

$|A| \leq |V| - |D| + 1$

Thus, $\gamma_{fa2}(G) \leq |V| - \gamma_{fa}(G) + 1$.

Theorem 19. For any fuzzy graph $G$, bounds of accurate 2-domination number is

$$\frac{|V|}{\Delta_{N_a}(G)} + 1 \leq \gamma_{fa2}(G) \leq \frac{|V|\Delta_{N_a}(G)}{\Delta_{N_a}(G) + 1} + 1$$

Proof. Let $G$ be a fuzzy graph and a subset $D \subseteq V$, be an accurate 2-dominating set of $G$. Since, $D$ is an accurate 2-dominating set of $G$, then each nodes in $V - D$ must have at least two strong neighbours in $D$.

Case (i): Let us prove that $\gamma_{fa2}(G) \geq \frac{|V|}{\Delta_{N_a}(G) + 1}$

If $D$ be an accurate dominating set of $G$ and there exists a node $v \in V$, which
has maximum strong neighbours in \( V(G) \), then \( v \) can dominate itself and number of nodes in \( \Delta_{N_S} \). Therefore, accurate domination number of fuzzy graph cannot be less than \( \frac{|V|}{\Delta_{N_S}} + 1 \). That is

\[
\gamma_{fa}(G) \geq \frac{|V|}{\Delta_{N_S}(G) + 1}
\]

Since \( \gamma_{fa}(G) \leq \gamma_{fa2}(G) \). Then,

\[
\gamma_{fa2}(G) \geq \frac{|V|}{\Delta_{N_S}(G) + 1}
\]

**Case (ii):** Now, let us prove, \( \gamma_{fa2}(G) \leq \frac{|V|\Delta_{N_S}(G)}{\Delta_{N_S}(G)+1} + 1 \).

By theorem 18, \( \gamma_{fa2}(G) \leq |V| - \gamma_{fa}(G) + 1 \)

By equation 1, \( \gamma_{fa2}(G) \leq |V| - \frac{|V|\Delta_{N_S}(G)}{\Delta_{N_S}(G)+1} + 1 \)

\[
\gamma_{fa2}(G) \leq \frac{|V|}{\Delta_{N_S}(G) + 1} \]

\[
\gamma_{fa2}(G) \leq \frac{|V|\Delta_{N_S}(G)}{\Delta_{N_S}(G)+1} + 1
\]

Therefore, by equation 2 and 3, we get \( \frac{|V|}{\Delta_{N_S}(G)+1} \leq \gamma_{fa2}(G) \leq \frac{|V|\Delta_{N_S}(G)}{\Delta_{N_S}(G)+1} + 1 \).

**Corollary 20.** For any fuzzy graph \( G \), \( \gamma_{fa2}(G) \geq \frac{|V|}{\Delta_{N_S}(G)+1} \).

**Proof.** Let \( G \) be a fuzzy graph and \( D \) be a minimum accurate 2-dominating set of fuzzy graph \( G \), where \( D \subseteq V \). Let \( S \) be the set of all strong arcs between the sets \( D \) and \( V - D \).

If there exist a node \( u \in V \), whose \( |dN_S(u)| = \Delta_{N_S}(G) \), then the strong neighbourhood degree of every node \( v \in D \) does not exceed \( \Delta_{N_S}(G) \). Therefore,

\[
S \leq \Delta_{N_S}(G) \cdot \gamma_{fa2}(G)
\]

And also since every node \( v \in V - D \) has atleast two strong neighbours in \( D \). Then,

\[
S \geq 2(|V| - \gamma_{fa2}(G))
\]

From equations 4 and 5,

\[
2(|V| - \gamma_{fa2}(G)) \leq \Delta_{N_S}(G) \cdot \gamma_{fa2}(G)
\]

\[
2|V| - 2\gamma_{fa2}(G) \leq \Delta_{N_S}(G) \cdot \gamma_{fa2}(G)
\]

\[
2|V| \leq \Delta_{N_S}(G) \cdot \gamma_{fa2}(G) + 2\gamma_{fa2}(G)
\]

\[
2|V| \leq (\Delta_{N_S}(G) + 2) \cdot \gamma_{fa2}(G)
\]

\[
\therefore \gamma_{fa2}(G) \geq \frac{|V|}{\Delta_{N_S}(G)+1}.
\]
4 Conclusion

In this paper, the accurate 2-dominating set and accurate 2-domination number are defined. Some results on accurate 2-dominating sets and accurate 2-domination number are discussed. Also, bounds of accurate 2-domination number of fuzzy graphs are found.

References


