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$\gamma_M(G)$-Independent Fixed Free and Totally Free Graphs

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Abstract

This paper deals with the study of fixed, free and totally free graphs with respect to minimum majority dominating sets of a graph G($\gamma_M(G)$). Then the notion of $\gamma_M(G)$- independent fixed, free and totally free graphs are introduced and some results on these parameters are established.

AMS Subject Classification: 05C69

Key Words and Phrases: Majority Dominating Set, Majority Domination Number -$\gamma_M(G)$, $\gamma_M(G)$-Independent sets.

1 Introduction

Prof. Bauer et al. [**1**] obtained some results in γ-fixed, γ-free and γ-totally free graphs. Then Prof. E.Sampathkumar et al. [**10**] introduced some properties of vertices classified according to whether they belong to all, at least one but not all, or none of the minimum dominating sets. Then G can be said to be t-fixed if an element e belongs to every t-set, t-free if an element e belongs to some t-set but not to all t-sets and t-totally free if e belongs to no t-set. In this article, graphs are classified into three categories, with respect to $\gamma_M(G)$-sets namely γ_M-independent fixed graph, γ_M-independent free graph and γ_M-independent totally free graph and for certain graphs these properties are investigated.

By a graph [4] $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The open neighborhood of v is defined to be the set of vertices adjacent to v in G, and is denoted as $N(v)$. Further, the closed neighborhood of v is defined by $N[v] = N(v) \cup \{v\}$. The closed neighborhood of a set of vertices S is denoted as $N[S] = \cup \lim_{s \in S} N[s]$.

Definition 1. *[5] A subset $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a Majority Dominating Set if at least half of the vertices of V are either in S or adjacent to elements of S.*
(i. e.) $|N[S]| \geq \left\lceil \frac{|V(G)|}{2} \right\rceil$.

Definition 2. **[5]** *A majority dominating set S is minimal if no proper subset of S is a majority dominating set. The minimum cardinality of a minimal majority dominating set is called the majority domination number of G and is denoted by $\gamma_M(G)$. The maximum cardinality of a minimal majority dominating set is called upper majority domination number of G and it is denoted by $\Gamma_M(G)$. Also, Majority dominating sets have super hereditary property.* ∎

2 Definitions and Examples

Definition 3. *Let G be a graph with p vertices and γ_M-set is a minimum majority dominating set. If every γ_M-set of a graph G is independent then the graph G is called a γ_M-independent fixed graph.* ∎

Definition 4. *If there exist some γ_M-sets of G which are independent and some γ_M-sets of G which are not independent then the graph G is called a γ_M-independent free graph.* ∎

Definition 5. *If every γ_M-set of G is not independent then the graph G is called a γ_M-independent totally free graph.* ∎

Proposition 6. **[5]** *Let $G = C_p$ be a cycle on p vertices, $p \geq 3$. Then $\gamma_M(G) = \lceil \frac{p}{6} \rceil$.*

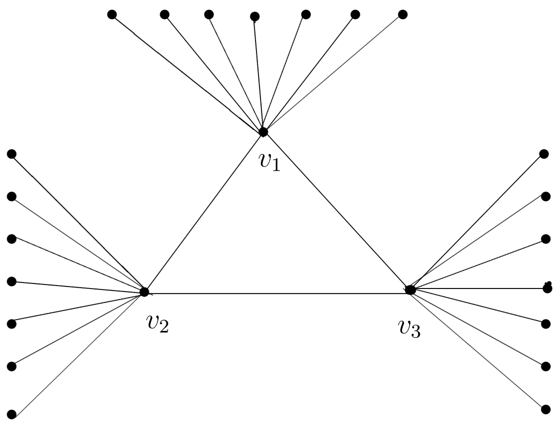
Example 7. *Let $G = C_9$ be a cycle on 9 vertices. Then by proposition [6], $\gamma_M(C_9) = 2$ and $D = \{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3, v_8\}, \{v_4, v_7\}, \{v_6, v_9\}\}$ are γ_M-sets of G. All γ_M-sets are independent. Therefore $G = C_9$ is a γ_M-independent fixed graph.*

Proposition 8. **[5]** *Let $G = P_p$ be a path on p vertices, $p \geq 2$. Then $\gamma_M(G) = \lceil \frac{p}{6} \rceil$.* ∎

Example 9. *Let $G = P_8$. By the above result, $\gamma_M(P_8) = 2$ and $D = \{\{v_1, v_3\}, \{v_2, v_4\}$,
$\{v_5, v_7\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\}$ are some γ_M-sets of G in which some are independent $(1-3)$ and some are not independent $(4-6)$. Therefore $G = P_8$ is a γ_M-independent free graph.* ∎

Example 10. *In the following graph G, $p = 24$ and $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}$ are the only γ_M-sets of G and all sets are not independent. Therefore this graph is* *called γ_M-independent totally free graph.*

$G:$



Proposition 11. **[6]** *Let G be any graph with p vertices. Then $\gamma_M(G) = 1$ if and only if there exists a vertex of degree v such that $d\,(v) \geq \lceil \frac{p}{2} \rceil - 1$.* ■

Proposition 12. *Let G be any graph with p vertices. $\gamma_M(G) = 2$ if and only if $\Delta(G) \leq \frac{p}{2}\rceil - 2$ and there exist two vertices u, v such that $d(u) + d(v) \geq \frac{p}{2}\rceil + d\,(u, v) - 2$ where $d\,(u, v) = |N[u] \cap N[v]|$.*

observation 13. • *If $\gamma_M(G) = 1$ then G is a γ_M-independent fixed graph.*

- *If $\gamma_M(G) = 1$ then G is a γ_M-independent fixed graph.*
- *If a Graph G has a full degree vertex then G is a γ_M- independent fixed graph. For example, $G = K_p$ and $G = F_n = P_{n-1} \vee K_1$, a fan.*
- *If a graph G has a majority dominating vertex then G is a γ_M- independent fixed graph.*
  For example, $G = D_{r,s}$, a a double star and $G = K_{m,n}$ a complete bibartite graph.

Proposition 14. *Let G be a γ_M-independent free graph with $\gamma_M(G) = 2$. Let $D = \{u, v\}$ be a γ_M-set of G which is not independent. Then D is a γ_M-independent set of $G - e$, where $e = uv$.*

Proof: Let G be a γ_M- independent free graph with $\gamma_M(G) = 2$.Since the γ_M-set $D = \{u, v\}$ is not independent, the vertices u and v are adjacent in G and$G' = G - e$ where $e = uv$. Then $d_{G'}(u) = d(u) - 1$ and $d_{G'}(v) = d(v) - 1$. Therefore $|N_{G'}[u] \cap N_{G'}[v]| = |N_G[u] \cap N_G[v]| - 2$. By proposition **??**, $d_{G'}(u) + d_{G'}(v) = d(u) + d(v) - 2 \geq \lceil \frac{p}{2} \rceil + |N_G[u] \cap N_G[v]| - 2 - 2$. Therefore $d_{G'}(u) + d_{G'}(v) \geq \lceil \frac{p}{2} \rceil + |N_{G'}[u] \cap N_{G'}[v]| - 2$. That implies $\gamma_M G' = 2$ and $D = \{u, v\}$ is a γ_M-independent set of $G - e$.

Proposition 15. *Let $D = \{u, v\}$ be a γ_M-set which is independent if $e \in \langle N[u] \vee N[v] \rangle$ then D is and independent γ_M -set of G.*

Result 16.
- *Let G be a γ_M-independent free graph with p vertices and $\gamma_M(G) = 2$. Let $D = \{u, v\}$ be any γ_M-set of G such that $|N[D]| = \lceil \frac{p}{2} \rceil$. If either $e \notin \langle N[u] \rangle$ or $e \notin \langle N[v] \rangle$ then D is also a $\gamma_M$set of $G - e$.*
- *Let G be a γ_M- independent free graph with p vertices. Let D be any independent or non-independent γ_M-set of G with $|N[D]| > \lceil \frac{p}{2} \rceil$. If either an appropriate edge $e \in \langle N[u] \bigcup N[v] \rangle = g(G)$ or $e \notin g(G)$, D is an independent or non- independent in $(G - e)$.*

Proposition 17. *Let G be a γ_M-independent free graph with $\gamma_M(G) = 2$. Let $D = \{u, v\}$ be γ_M-set of G. Let $e \notin \langle N[u] \cup N[v] \rangle$ or $e \notin \langle N[u] \rangle$ and if $e = ux$, then $x \in N[v]$ or $e \in \langle N[v] \rangle$ and if $e = vx$, then $x \in N[u]$. Then D is γ_M-set of $G - e$.*

Proof : Let G be a γ_M-independent free graph. Then G has both γ_M-independent and γ_M-non independent sets.
Case(i) : Let $e \notin \langle N[u] \cup N[v] \rangle$. Then $d_G(u) = d_{G'}(u)$ and $d_G(v) = d_{G'}(v)$. $|N_G[u] \cap N_G[v]| = |N_{G'}[u] \cap N_{G'}[v]|$. Then $d_{G'}(u) + d_{G'}(v) = d_G(u) + d_G(v) \geq \lceil \frac{p}{2} \rceil + |N_G[u] \cap N_G[v]| - 2$. This implies that $d_{G'}(u) + d_{G'}(v) \geq \lceil \frac{p}{2} \rceil + |N_{G'}[u] \cap N_{G'}[v]| - 2$. Therefore $D = \{u, v\}$ is an γ_M-set of $G' = G - e$.
Case(ii) : Let $e = ux$ and $x \in N[v]$. Then $d_{G'}(u) = d_G(u) - 1$ and $d_G(v) = d_{G'}(v)$. Therefore $|N_{G'}[u] \cap N_{G'}[v]| = |N_G[u] \cap N_G[v]| - 1$.
Then $d_{G'}(u) + d_{G'}(v) = d_G(u) + d_G(v) - 1 \geq \lceil \frac{p}{2} \rceil + |N_G[u] \cap N_G[v]| - 1 - 2$. Then $d_{G'}(u) + d_{G'}(v) \geq \lceil \frac{p}{2} \rceil + |N_{G'}[u] \cap N_{G'}[v]| - 2$. Therefore $D = \{u, v\}$ is an γ_M-set of $G' = G - e$.
Case(iii) : If $e = vx$ and $x \in N[u]$, then proceeding as above we get that $D = \{u, v\}$ is an γ_M-set of $G' = G - e$. ■

Remerk 18. *Let D be a γ_M-set with $|N[D]| = \lceil \frac{p}{2} \rceil$. Then there are graphs G with $\gamma_M(G) = 2$ and in which there are γ_M-sets of G which are not majority dominating set in $G - e$ for a suitable $e \in E(G)$.*

Proposition 19. *Let G be a γ_M-independent free graph with $\gamma_M(G) = 2$. Let G have two disjoint non independent γ_M-sets. Then there exists an edge $e \in E(G)$ such that $G - e$ is a γ_M-independent free graph.*

Proof: Let $D = \{u, v\}$ be a γ_M-set of G which is not independent. Let $e = uv$ Then in $G - e$, D is a γ_M-set which is independent. If G has another γ_M-set D' which is not independent then D' is an γ_M-set of $G - e$ and hence $G - e$ has an independent γ_M-set D and an non independent γ_M-set D'. ■

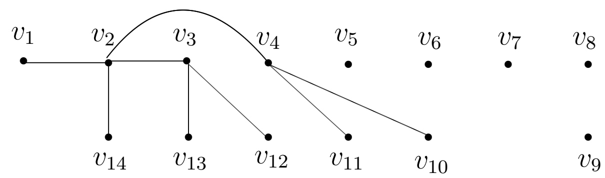
Proposition 20. *If G is a γ_M-independent free graph with unique non-independent γ_M-set D then $G^{'=(G-e)}$ is a γ_M-independent fixed graph if $e = u_1u_2 \in D$.*

Proof Let $D = \{u_1, u_2, u_3, \ldots u_m\}$ be a unique non-independent γ_M-set of G

with $|N[D]| \geq \lceil \frac{p}{2} \rceil$. Then D contains atleast two vertices which are adjacent say $u_1, u_2 \in D$. Let $e = (u_1, u_2) \in D$. Then $N[u_1] \bigcap N[u_2] \neq \Rightarrow \; d\,(u_1, u_2) = 1$.
Consider the graph $G' = G - e$. Then $d_{G'}(u_1) = \mathrm{d(u_1)} - 1$ and $d_{G'}(u_2) = d(u_2) - 1$. Also, $|N[u_1] \bigcap N[u_2]|_{G'} = |N[u_1] \bigcap N[u_2]|_G - 2$. It implies that $d_{G'(u_1,u_2)>1}$ but the vertices u_1 and u_2 are dominated by itself in G'. Therefore D is an independent set in G^1.
Claim: D is a γ_M-set of G'.
If u_1 and u_2 are adjacent in G then $|N[D]|_G = \sum_{u_i \in D} d(u_i) + |D|$.
Since $|N[D]|_G \geq \lceil \frac{p}{2} \rceil \Rightarrow \left(\sum_{u_i \in D} d(u_i) + |D| - 2\right) \geq \lceil \frac{p}{2} \rceil \Rightarrow (\sum d(u_i) + |D| \geq \lceil \frac{p}{2} \rceil + 2$ $N[D]_{G' > \lceil \frac{p}{2} \rceil}$. Therefore D is a γ_M set in G'. Hence D is a γ_M independent set in G'. Thus the given graph G becomes γ_M- independent fixed graph.

Remerk 21. *There exists a graph G which is γ_M-independent free graph and for any edge $e \in E(G)$, $\;G - e$ is not a γ_M-independent free graph.*

G :



In graph G, $\{v_2, v_3\}$, $\{v_2, v_4\}$ are γ_M-sets which are not independent and $\{v_3, v_4\}$ is the only γ_M- independent set. For any edge $e \in E(G)$, $\;G - e$ has γ_M-sets which are only independent. γ_M-sets are not majority ddominating sets of $(G - e)$. For any edge $e \in E(G)$, $\;G - e$ is not a γ_M-independent free graph. ∎

Proposition 22. *Let $G = K_p,\; K_{1,p-1},\; p \geq 2,\; \overline{K_p},\; W_p,\; p \geq 5$. Then G is γ_M-independent fixed graph.* ∎

Proposition 23. *If G has a unique non-independent γ_M-set D then $G' = (G - u)$ has atleast one independent γ_M-set when $u, v \in D$.*

proof The proof is obvious. If G has a unique non-independent γ_M-set then $G' = (G - uv)$ has also a unique non-independent γ_M-set when $u, v \notin D$.

3 Characterisation Theorems on γ_M-independent fixed and free graphs

Let P_p be a path, where $p \geq 2$.

- P_p is γ_M-independent fixed graph if and only if $p \equiv 0, 3, 4, 5 \pmod 6$
- P_p is γ_M-independent free graph if and only if $p \equiv 1, 2 \pmod 6$.

Proof : Let P_p be a path and $V(P_p) = \{u_1, u_2, ...u_p\}$. Assume that P_p is γ_M - independent fixed graph. $\gamma_M(P_p) = \left\lceil \frac{p}{6} \right\rceil$. Let $p = 6m + l,\; 0 \leq l \leq 5$.
Case (i): Let $l \neq 0$. Consider $V(P_{p-l}) = \{u_1, u_2, \cdots, u_{p-l}\}$. Then $D_1 = \{u_2, u_5, u_8, ..., u_{\left(\frac{p-l-2}{2}\right)}\}$ is a γ_M-set of P_{p-l}. Therefore $|D_1| = \frac{p-l}{6} = m$. Then $|N[D_1]| = \frac{p-l}{2} = 3m$.

Sub case (i): Let $l = 1$ (or) 2.
Then $V(P_p) - N[D_1] \neq \phi$. Let $u \in V(P_p) - N[D_1]$. $|N[D_1 \cup \{u\}]| \geq 3m + 1$ and $|D_1 + \{u\}| = m + 1 = \lceil \frac{p}{6} \rceil = \gamma_M(P_p)$. Therefore $D_1 + \{u\}$ is a γ_M-set of P_p for any $u \in V(P_p) - N[D_1]$. Hence there exists some γ_M-set which is independent and some γ_M-set which is not independent when $p \equiv 1$ or 2 (mod 6). Therefore P_p is γ_M-independent free graph if $p \equiv 1$ or 2 (mod 6).
Sub case (ii): Let $l = 3$ (or) 4. Let D be a γ_M-set of P_p. $|D| = \lceil \frac{p}{6} \rceil$.
Claim : D is independent.
Suppose $uv \in E(G)$ for some $u, v \in D$. For every $x, y \in D - \{u, v\}$, $d(x, y) \geq 3$. Then $|N[D - \{u, v\}]| = 3(\lceil \frac{p}{6} \rceil - 2) = 3((m + 1) - 2) = \frac{p-l}{2} - 3$. When $l = 3$, $\frac{p-l}{2} - 3 < \lceil \frac{p}{2} \rceil - 4$. When $l = 4$, $\frac{p-l}{2} - 3 < \lceil \frac{p}{2} \rceil - 4$. Therefore $|N[D - \{u, v\}]| < \lceil \frac{p}{2} \rceil - 4$. Since $\{u, v\}$ dominates 4 vertices, $N[D] < (\frac{p}{2})$, a contradiction to D is a γ_M-set. Hence D is an γ_M- independent set for G_1 if $p \cong 3, 4\, (mod 6)$.
Sub case(iii): Let $l = 5$. Suppose D is not independent. Then there exists $u, v \in D$ such that $u, v \in E(G)$. With out loss of generality, assume that $d(x, y) \geq 3$ for every $x, y \in D - \{u, v\}$.
Claim : D is independent.
Suppose $uv \in E(G)$ for some $u, v \in D$. For every $x, y \in D - \{u, v\}$, $d(x, y) \geq 3$. Then $|N[D - \{u, v\}]| = \frac{p-l}{2} - 3$. When $l = 5$, $|N[D - \{u, v\}]| < \lceil \frac{p}{2} \rceil - 4$. Applying the same argument, D is anindependent γ_M-set for G if $p \cong 5\, (mod 6)$.
Case (ii): Let $l = 0$. Let D be a γ_M-set of P_p. Then $|D| = \frac{p}{6}$. Suppose that $N[u] \cap N[v] \neq \phi$ for some $u, v \in D$. Then $|N[D]| \leq \sum_{w \neq u,v} |N[w]| + |N[u] \cup N[v]| \leq (|D| - 2)3 + 5 = \frac{p}{2} - 1$ which is contradiction to D is a γ_M-set. Hence D is an independent γ_M-set for G, if $p \cong 0\, (mod 6)$. Thu;s the graph $G = P_p$ is γ_M-independent fixed graph if $p \cong 0, 3, 4, 5\, (mod 6)$. ■

Let C_p be a cycle of order p, $p \geq 6$. Then,

- C_p is γ_M -independent fixed graph if and only if $p \equiv 0, 3, 4, 5$ (mod 6).
- C_p is γ_M-independent free graph if and only if $p \equiv 1, 2$ (mod 6).

[9] Let C be a caterpillar with exactly one pendent edge at each internal vertex, $p > 5$. Then $\gamma_M(C) = \lceil \frac{p}{8} \rceil$ ■

[9] Let C be a caterpillar of order p with exactly k pendent edges at each internal vertex. Then $\gamma_M(C) = \left\lceil \frac{p}{2(k+3)} \right\rceil$. ■

Let C be a caterpillar with exactly one pendent edge at each internal vertex, $p > 5$. Then C is γ_M-independent fixed graph if $p \equiv 0, 5, 6, 7(mod\ 8)$ and C is γ_M-independent free graph if $p \equiv 1, 2, 3, 4(mod\ 8)$. **Proof:** Let C be a caterpillar with exactly one pendent edge at each internal vertex. $|D| = \lceil \frac{p}{8} \rceil$
Case(i) : Let $p \equiv 0, 5, 6, 7(mod\ 8)$. Then any γ_M-set cannot contain consecutive vertices. Let D be an γ_M-set which contains v_1, v_2 on the spine which are adjacent.
Sub case(i) : Let $p = 8m$. Then v_1, v_2 together can dominate at most six vertices. Assuming that all other vertices of $d(u, v) \geq 3$ which dominate four vertices each $|N[D]| \leq \frac{p}{2} - 2$, which is a contradiction to the γ_M-set of G.
Sub case(ii) : Let $p = 8m + l\ where\ l = 5, 6, 7$. Then $|N[D]| \leq \frac{p}{2} + 1 - 2 = \frac{p}{2} - 1$, a contradiction . Therefore every γ_M-set is independent and C is γ_M-independent

fixed graph.
Case(ii) : Let $p \equiv 1, 2, 3, 4(mod\ 8)$. Then there exists a γ_M-sets which are independent and γ_M-sets which are not independent. Let the spine of C be $\{v_1, v_2, ..., v_{\frac{p}{2}}\}$. Let $D = \{v_2, v_5, v_8, ..., v_{(3\lceil\frac{p}{8}\rceil-1)}\}$. Then D is independent majority dominating set and $|D| = 3\left\lceil\frac{p}{8}\right\rceil = \gamma_M$. Therefore there exists a γ_M-set which is independent. Let $D_1 = \{v_2, v_3, v_6, v_9, ..., v_{(3\lceil\frac{p}{8}\rceil-1)}\}$.
$|N[D]| = 4(\left\lceil\frac{p}{8}\right\rceil - 1) + 2 = \frac{p}{2} + 1 \quad if\ p \equiv 1, 2(mod\ 8)$ and $|N[D]| = \frac{p}{2} \quad if\ p \equiv 3, 4(mod\ 8)$.
Therefore D_1 is a γ_M-set which is not independent. Since v_2 and v_3 are adjacent in C. Hence C is a γ_M-independent free graph. ■

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