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Solving Integer Linear Programming Problems With Pentagonal Fuzzy Numbers

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Abstract

A distinct decomposition method for solving integer linear programming problems with pentagonal fuzzy numbers by using classical integer linear programming has been proposed. Some properties of the above said notion have been studied. Both the symmetric and non-symmetric fuzzy numbers have been utilized in integer linear programming problem. Relevant numerical illustrations have been included to justify the proposed notion.

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1 Introduction

The concept linear programming problem is to find out the best solution to the real world problems where the available information are not exact or not precise. In that situation linear programming model helps lot. Firstly, the concept Fuzzy linear programming was proposed by Tanaka et al[11]. It play a vital role in Fuzzy modeling, which can formulate the uncertainty. Nasseri [9] has proposed a new method for solving the Fuzzy linear programming problems in which he has used the fuzzy ranking method for converting the fuzzy objective function into crisp objective function. Fuzzy linear programming was studied by many researchers[11, 9, 4, 3, 2, 6, 7, 8, 12].

Herrera and Verdegay [5] have proposed three methods for solving three models of Fuzzy integer linear programming. Allahviranloo et al[1] discussed a model of Fuzzy integer linear programming problem with fuzzy variable and proposed a new method for solving. Pandian and Jayalakshmi [10] have proposed a decomposition method for solving Fuzzy integer linear programming problem with fuzzy variables by using classical integer linear programming.

In this paper, section 2 contains some of the basic definitions and theorem. In section 3, fuzzy integer linear programming with fuzzy variables are discussed. A unique definition has been proposed to make parametric form of the fuzzy number. In section 4, relevant numerical illustrations are given. Finally conclusion is included in section 5.

2 Preliminaries

Definition 1. A Fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(x, \mu_{A(x)}) : x \in A, \mu_{A(x)} \in [0,1]\}$. In the pair $(x, \mu_{A(x)})$, the first element x belong to the classical set A, the second element $\mu_{A(x)}$, belong to the interval $[0, 1]$, called Membership function.

Definition 2. A Pentagonal fuzzy number $\tilde{A}_P = (a_1, a_2, a_3, a_4, a_5)$ where a_1, a_2, a_3, a_4, a_5 are real numbers and its membership is given below

$$\mu_A(x) = \begin{cases} 0 & , x < a_1 \\ \left[\frac{x-a_1}{a_2-a_1}\right] & , a_1 \le x \le a_2 \\ \left[\frac{x-a_2}{a_3-a_2}\right] & , a_2 \le x \le a_3 \\ 1 & , x = a_3 \\ \left[\frac{a_4-x}{a_4-a_3}\right] & , a_3 \le x \le a_4 \\ \left[\frac{a_5-x}{a_5-a_4}\right] & , a_4 \le x \le a_5 \\ 0 & , x > a_5 \end{cases}$$

Definition 3. Let $\tilde{A}_P = (a_1, a_2, a_3, a_4, a_5)$ and $\tilde{B}_P = (b_1, b_2, b_3, b_4, b_5)$ be two pentagonal fuzzy numbers then,

(i) **Addition** :
$\tilde{A}_p + \tilde{B}_p = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5)$.

(ii) **Subtraction** :
$\tilde{A}_p - \tilde{B}_p = (a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4, a_5 - b_5)$.

(iii) **ScalarMultiplication** :
If $k \neq 0$ is scalar $k\tilde{A}_p$ is defined as

$$k\tilde{A}_p = \begin{cases} (ka_1, ka_2, ka_3, ka_4, ka_5) & \text{if } k \ge 0 \\ (ka_5, ka_4, ka_3, ka_2, ka_1) & \text{if } k < 0 \end{cases}$$

Definition 4. Let $\tilde{A}_P$ and $\tilde{B}_P$ be in *F(R)*. Then,

(i) $\tilde{A}_P = \tilde{B}_P \iff a_i = b_i,\ for\ all\ i = 1\ to\ 5$

(ii) $\tilde{A}_P \le \tilde{B}_P \iff a_i \le b_i,\ for\ all\ i = 1\ to\ 5$.

Definition 5. Let $\tilde{A}_P$ be in $F(R)$. Then,

(i) $\tilde{A}_P$ is said to be positive if $a_i \geq 0$ for all $i = 1$ *to* 5
(ii) $\tilde{A}_P$ is said to be integer if $a_i \geq 0$ for all $i = 1$ *to* 5 and are integers.

Definition 6. A real fuzzy vector $\tilde{b} = (\tilde{b})_{m\times 1}$ is called non-negative and denoted by $\tilde{b} \geq 0$, if each element of $\tilde{b}$ is a non-negative real fuzzy number, that is $\tilde{b} \geq 0, i = 1, 2, , m$.

Consider the following $m \times n$ fuzzy linear system with non-negative real pentagonal fuzzy numbers

$$A\tilde{x} \leq \tilde{b}$$

Where $A = (a_{ij})_{m\times n}$ is a non-negative crisp matrix and $\tilde{x} = (\tilde{x}_j), \tilde{b} = (\tilde{b}_i)$ are non-negative fuzzy vectors and $\tilde{x}_j, \tilde{b}_i \in F(R)$, for all $1 \leq j \leq n$ and $1 \leq i \leq m$.

Definition 7. A non-negative fuzzy vector $\tilde{x}$ is said to be a solution of the fuzzy linear system $A\tilde{x} \leq \tilde{b}$ if $\tilde{x}$ satisfies equation $A\tilde{x} \leq \tilde{b}$.
Using the definition 4 and 7 and the arithmetic operations on pentagonal fuzzy number, we obtain the following theorem.

Theorem 8. *Let* $A\tilde{x} \leq \tilde{b}$ *be an* $m \times n$ *fuzzy linear system where* $A = (a_{ij})_{m\times n}$*is a non-negative crisp matrix ,* $\tilde{x} = (\tilde{x}_j)$*,*$\tilde{b} = (\tilde{b}_i)$ *are non-negative real pentagonal fuzzy vectors and* $\tilde{x}_j = (x_1^j, x_2^j, x_3^j, x_4^j, x_5^j)$ *and* $\tilde{b}_i = (b_1^i, b_2^i, b_3^i, b_4^i, b_5^i) \in F(R)$ *for all* $1 \leq j \leq n; 1 \leq i \leq m$*. If* $x_3^\circ = (x_3^{j\circ})_{n\times 1}$ *is a solution of the system* $Ax_3 \leq b_3$*,*$x_3 \geq 0$ *where* $x_3 = (x_3^j)_{n\times 1}$ *and* $b_3 = (b_3^i)_{m\times 1}$*,* $x_2^\circ = (x_2^{j^\circ})_{n\times 1}$ *is a solution of the system* $Ax_2 \leq b_2$*,*$x_2 \geq 0$*,*$x_2 - x_3^\circ \leq 0$ *where* $x_2 = (x_2^j)_{n\times 1}$ *and* $b_2 = (b_2^i)_{m\times 1}$*,and* $x_1^\circ = (x_1^{j^\circ})_{n\times 1}$ *is a solution of the system* $Ax_1 \leq b_1$*,* $x_1 \geq 0$*,* $x_1 - x_3^\circ \leq 0$ *where* $x_1 = (x_1^j)_{n\times 1}$ *and* $b_1 = (b_1^i)_{m\times 1}$*, and* $x_4^\circ = (x_4^{j^\circ})_{n\times 1}$ *is a solution of the system* $Ax_4 \leq b_4$*,* $x_4 \geq 0$*,* $x_4 - x_3^\circ \geq 0$ *where* $x_4 = (x_4^j)_{n\times 1}$ *and* $b_4 = (b_4^i)_{m\times 1}$*,and* $x_5^\circ = (x_5^{j^\circ})_{n\times 1}$ *is a solution of the system* $Ax_5 \leq b_5, x_5 \geq 0$ *,* $x_5 - x_3^\circ \geq 0$*, where* $x_5 = (x_5^j)_{n\times 1}$ *and* $b_5 = (b_5^i)_{m\times 1}$ *then* $\tilde{x}^\circ = (\tilde{x}_j^\circ)$ *is a solution of the system* $A\tilde{x} \leq \tilde{b}$ *where* $\tilde{x}_j^\circ = \left(\tilde{x}_1^{j^\circ}, \tilde{x}_2^{j^\circ}, \tilde{x}_3^{j^\circ}, \tilde{x}_4^{j^\circ}, \tilde{x}_5^{j^\circ}\right)$

3 Fuzzy Integer Linear Programming

Consider the following integer linear programming problem with fuzzy variables $Max\ \tilde{z} = c\tilde{x}$ subject to $A\tilde{x} \leq \tilde{b}$, $\tilde{x} \geq 0$ and are integers. Where the coefficient matrix $A = (a_{ij})_{m\times n}$ is a non-negative real crisp matrix and the cost vector $c = (c_1, c_2, c_3, \ldots, c_n)$ is non-negative crisp matrix,$\tilde{x} = (\tilde{x}_j)_{n\times 1}, \tilde{b} = (\tilde{b}_i)_{m\times 1}$ are non-negative real fuzzy vector such that $\tilde{x}_j, \tilde{b}_i \in F(R)$ for all $1 \leq j \leq n$ and $1 \leq i \leq m$.

Definition 9. A fuzzy vector $\tilde{x}$ is said to be a feasible solution of the problem if $\tilde{x}$ satisfies $A\tilde{x} \leq \tilde{b}$, $\tilde{x} \geq 0$.

Definition 10. A feasible solution $\tilde{x}$ of the problem is said to be an optimal solution of the problem if there exist no feasible $\tilde{u} = (\tilde{u}_j)_{n\times 1}$ of the problem such that $c\tilde{u} > c\tilde{x}$.

Theorem 11. *A fuzzy vector* $\tilde{x}^\circ = (\tilde{x}_1^\circ, \tilde{x}_2^\circ, \tilde{x}_3^\circ, \tilde{x}_4^\circ, \tilde{x}_5^\circ)$ *is an optimal solution of the problem iff* $\tilde{x}_3^\circ, \tilde{x}_2^\circ, \tilde{x}_1^\circ, \tilde{x}_4^\circ$ *and* $\tilde{x}_5^\circ$ *are optimal solution of the following crisp integer linear programming problems (iii),(ii),(i),(iv), and (v) respectively where*

(iii). Maximize $z_3 = cx_3$ *, subject to* $Ax_3 \leq b_3$ *,*$x_3 \geq 0$ *are integers.*

(ii). Maximize $z_2 = cx_2$ *, subject to* $Ax_2 \leq b_2$ *,*$x_2 \geq 0$*,* $x_2 \leq x_3^\circ$ *are integers.*

(i). Maximize $z_1 = cx_1$ *, subject to* $Ax_1 \leq b_1$ *,*$x_1 \geq 0$ *,* $x_1 \leq x_3^\circ$ *are integers.*

(iv). Maximize $z_4 = cx_4$ *, subject to* $Ax_4 \leq b_4$ *,* $x_4 \geq 0$ *,* $x_4 \geq x_3^\circ$ *are integers.*

(v). Maximize $z_5 = cx_5$ *, subject to* $Ax_5 \leq b_5$*,* $x_5 \geq 0$ *,* $x_5 \geq x_3^\circ$ *are integers.*

Proof. Suppose that $\tilde{x}^\circ = (\tilde{x}_1^\circ, \tilde{x}_2^\circ, \tilde{x}_3^\circ, \tilde{x}_4^\circ, \tilde{x}_5^\circ)$ is an optimal solution of the problem.

Let $\tilde{x} = (x_1, x_2, x_3, x_4, x_5)$ be a feasible solution of the problem.
This implies that,
$c_1x_1 \leq c_1x_1^\circ; c_2x_2 \leq c_2x_2^\circ; c_3x_3 \leq c_3x_3^\circ; c_4x_4 \leq c_4x_4^\circ; c_5x_5 \leq c_5x_5^\circ$

$$Ax_1^\circ \leq b_1; Ax_2^\circ \leq b_2; Ax_3^\circ \leq b_3; Ax_4^\circ \leq b_4; Ax_5^\circ \leq b_5; x_1^\circ, x_2^\circ, x_3^\circ, x_4^\circ, x_5^\circ \geq 0$$

Let $\tilde{z} = (z_1, z_2, z_3, z_4, z_5)$ be the objective function of the problem.
Now, from the above equation, we have

$$Max\ z_1 = c_1x_1^\circ; Max\ z_2 = c_2x_2^\circ; Max\ z_3 = c_3x_3^\circ; Max\ z_4 = c_4x_4^\circ; Max\ z_5 = c_5x_5^\circ$$

Therefore, we conclude that $x_3^\circ, x_2^\circ, x_1^\circ, x_4^\circ$, and x_5° are optimal solution of the crisp integer linear programming problems (iii),(ii),(i),(iv) and (v).

Suppose that $x_3^\circ, x_2^\circ, x_1^\circ, x_4^\circ$, and x_5° are optimal solution of the following crisp integer linear programming problems (iii),(ii),(i),(iv) and (v) with optimal values $z_3^\circ, z_2^\circ, z_1^\circ, z_4^\circ$ and z_5° respectively. This implies that $\tilde{x}^\circ = (x_1^\circ, x_2^\circ, x_3^\circ, x_4^\circ, x_5^\circ)$ is an optimal solution of the problem with optimal values $\tilde{z}^\circ = (z_1^\circ, z_2^\circ, z_3^\circ, z_4^\circ, z_5^\circ)$. Hence the theorem. □

Definition 12. A pentagonal fuzzy number is $\tilde{A}_P = (a_1, a_2, a_3, a_4, a_5)$ is said to have parametric representation if it satisfies the condition that $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$, for all $x_i^{'}s$ are real. Otherwise, the following two conditions can be employed to make them to parametric form.
(i). $x_4^\circ = x_3^\circ + (x_3^\circ - x_2^\circ)$.
(ii). $x_5^\circ = x_4^\circ + (x_4^\circ - x_3^\circ)$.
For example, if $\tilde{A} = (2, 4, 7, 11, 10)$ then the parametric form is $(2, 4, 7, 11, 12)$.

Theorem 13. *A fuzzy vector* $\tilde{x}^\circ = (\tilde{x}_1^\circ, \tilde{x}_2^\circ, \tilde{x}_3^\circ, \tilde{x}_4^\circ, \tilde{x}_5^\circ)$ *is an optimal solution of the problem iff* $\tilde{x}_3^\circ, \tilde{x}_2^\circ, \tilde{x}_1^\circ, \tilde{x}_4^\circ$ *and* $\tilde{x}_5^\circ$ *are optimal solution of the following crisp integer linear programming problems (iii),(ii),(i),(iv), and (v) respectively where,*

(iii). Maximize $z_3 = cx_3$ *, subject to* $Ax_3 \leq b_3$ *,*$x_3 \geq 0$ *are integers.*

(ii). Maximize $z_2 = cx_2$ *, subject to* $Ax_2 \leq b_2$ *,*$x_2 \geq 0$*,* $x_1 + x_2 \leq x_3^\circ$ *are integers.*

(i). Maximize $z_1 = cx_1$ *, subject to* $Ax_1 \leq b_1$ *,*$x_1 \geq 0$ *,* $x_1 + x_2 \leq x_3^{\circ}$ *are integers.*

(iv). Maximize $z_4 = cx_4$ *, subject to* $Ax_4 \leq b_4$ *,* $x_4 \geq 0$ *,* $x_1 + x_2 \geq x_3^{\circ}$ *are integers.*

(v). Maximize $z_5 = cx_5$ *, subject to* $Ax_5 \leq b_5$*,* $x_5 \geq 0$ *,* $x_1 + x_2 \geq x_3^{\circ}$ *are integers.*

4 Numerical Examples

Example 14. *Consider the following integer linear programming problem with fuzzy variables,* $Max\ \tilde{z} = 5\tilde{x}_1 + 15\tilde{x}_2$
subject to $4\tilde{x}_1 + 8\tilde{x}_2 \leq (28, 44, 65, 68, 73)$
$\tilde{x}_1 + 7\tilde{x}_2 \leq (18, 22, 33, 38, 46)$
,$\tilde{x}_1, \tilde{x}_2 \geq 0$ *and are integers.*
Let $\tilde{z} = (z_1, z_2, z_3, z_4, z_5)$ *,* $\tilde{x}_1 = (u_1, v_1, x_1, y_1, t_1)$ *and* $\tilde{x}_2 = (u_2, v_2, x_2, y_2, t_2)$
By using theorem 2, the fuzzy integer linear programming problem is converted into crisp integer linear programming problem becomes:

(iii). $Max\ z_3 = 5x_1 + 15x_2$ *subject to* $4x_1 + 8x_2 \leq 65; x_1 + 7x_2 \leq 33; x_1, x_2 \geq 0$ *and are integers.*

(ii). $Max\ z_2 = 5v_1 + 15v_2$ *subject to* $4v_1 + 8v_2 \leq 44; v_1 + 7v_2 \leq 22; v_1 \leq 10; v_2 \leq 3; v_1, v_2 \geq 0$ *and are integers.*

(i). $Max\ z_1 = 5u_1 + 15u_2$ *subject to* $4u_1 + 8u_2 \leq 28; u_1 + 7u_2 \leq 18; u_1 \leq 10; u_2 \leq 3; u_1, u_2 \geq 0$ *and are integers.*

(iv). $Max\ z_4 = 5y_1 + 15y_2$ *subject to* $4y_1 + 8y_2 \leq 68; y_1 + 7y_2 \leq 38; y_1 \geq 10; y_2 \geq 3; y_1,\ y_2 \geq 0$ *and are integers.*

(v). $Max\ z_5 = 5t_1 + 15t_2$ *subject to* $4t_1 + 8t_2 \leq 65; t_1 + 7t_2 \leq 33; t_1 \geq 10; t_2 \geq 3; t_1, t_2 \geq 0$ *and are integers.*

Now using an algorithm for integer linear programming problem and by definition 10, the solution of the fuzzy integer linear programming problem is

$$\tilde{x}_1 = (u_1, v_1, x_1, y_1, t_1) = (3, 7, 10, 11, 12)\,;$$
$$\tilde{x}_2 = (u_2, v_2, x_2, y_2, t_2) = (2, 2, 3, 3, 4)\,;\ and$$
$$\tilde{z} = (z_1, z_2, z_3, z_4, z_5) = (45, 65, 95, 100, 120).$$

Example 15. *Consider the following integer linear programming problem with fuzzy variables,*
$Max\ \tilde{z} = 5\tilde{x}_1 + 15\tilde{x}_2$
subject to $4\tilde{x}_1 + 8\tilde{x}_2 \leq (28, 44, 65, 68, 73)\,;$
$\tilde{x}_1 + 7\tilde{x}_2 \leq (18, 22, 33, 38, 46)\,, \tilde{x}_1, \tilde{x}_2 \geq 0$ *and are integers.*
Let $\tilde{z} = (z_1, z_2, z_3, z_4, z_5)\,, \tilde{x}_1 = (u_1, v_1, x_1, y_1, t_1)$ *and* $\tilde{x}_2 = (u_2, v_2, x_2, y_2, t_2)$.
By using theorem 3, the fuzzy integer linear programming problem is converted into crisp integer linear programming problem becomes:

(iii). *Max* $z_3 = 5x_1 + 15x_2$ *subject to* $4x_1 + 8x_2 \leq 65; x_1 + 7x_2 \leq 33; x_1, x_2 \geq 0$ *and are integers.*

(ii). *Max* $z_2 = 5v_1 + 15v_2$ *subject to* $4v_1 + 8v_2 \leq 44; v_1 + 7v_2 \leq 22; v_1 + v_2 \leq 13; v_1, v_2 \geq 0$ *and are integers.*

(i). *Max* $z_1 = 5u_1 + 15u_2$ *subject to* $4u_1 + 8u_2 \leq 28; u_1 + 7u_2 \leq 18; u_1 + u_2 \leq 13; u_1, u_2 \geq 0$ *and are integers.*

(iv). *Max* $z_4 = 5y_1 + 15y_2$ *subject to* $4y_1 + 8y_2 \leq 68; y_1 + 7y_2 \leq 38; y_1 + y_2 \geq 13; y_1, y_2 \geq 0$ *and are integers.*

(v). *Max* $z_5 = 5t_1 + 15t_2$ *subject to* $4t_1 + 8t_2 \leq 65; t_1 + 7t_2 \leq 33; t_1 + t_2 \geq 13; t_1, t_2 \geq 0$ *and are integers.*

Now using an algorithm for integer linear programming problem and by definition 10, the solution of the fuzzy integer linear programming problem is

$$\begin{gathered}\tilde{x}_1 = (u_1, v_1, x_1, y_1, t_1) = (3, 7, 10, 11, 12) \\ \tilde{x}_2 = (u_2, v_2, x_2, y_{2,2}) = (2, 2, 3, 3, 4) \text{ and} \\ \tilde{z} = (z_1, z_2, z_3, z_4, z_5) = (45, 65, 95, 100, 120).\end{gathered}$$

5 Conclusion

The Decomposition method provides an optimal solution to fuzzy integer linear programming problems without using the ranking functions and applying classical integer linear programming. In this work fuzzy parametric pentagonal numbers are utilized to solve the fuzzy integer linear programming problems. A unique definition has been proposed to make parametric form of the fuzzy number. This notion can be extended to some other optimization problems in future.

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