**International Journal of Pure and Applied Mathematics**
**Volume 118 No. 6 2018, 139-150**
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
**url:** http://www.ijpam.eu
Special Issue
ijpam.eu

# Fixed Point theorems for almost generalized $C$-contractive mappings in ordered complete partial metric spaces

A.Jennie Sebasty Pritha[1] and Dr.U.Karuppiah[2]

[1]*Department of Mathematics, Holy Cross College (Autonomous), Trichy-620 002.*
*jennipretha@gmail.com*
[2]*Department of Mathematics, St.Joseph's College (Autonomous), Trichy-620 002.*
*u.karuppiah@gmail.com*

**Abstract**

The purpose of this paper is to present some fixed point and common fixed point theorems for almost generalized $C$-contractive mappings in an ordered complete partial metric space. Finally an example is given to support our result..

**AMS Subject Classification:** 54H25; 47H10; 54E50.

**Key Words and Phrases:** Altering distance function,common fixed point, $C$-contractive mapping, fixed point, weakly increasing functions, partial metric space, ordered complete partial metric space.

## 1 Introduction

Ciric et al[1] introduced the concept of almost generalized contractive condition on mappings and proved some existential theorems on fixed points of such mappings in an ordered complete metric space. Shatanawi and Al-Rawashdeh[2] introduced the notion of an almost generalized $(\psi, \phi)$- contractive mapping in ordered metric spaces and established some fixed point and common fixed point results for such a mapping, where $\psi$ and $\phi$ are altering distance functions.The notion of partial metric space was introduced by Mathews[5] in 1994. The partial metric space is a generalization of the usual metric space in which $d(x, x)$ is no longer necessarily zero.

The purpose of this paper is to prove some fixed point and common fixed point theorems for the almost generalized $C$-contractive mappings in an ordered complete partial metric space. Specially, under suitable conditions, we show that if the fixed point set of such mappings is totally ordered, then it is singleton.In the end, an example is given to support the usability of our result.

## 2 Preliminaries

We first review the needed definitions. Throughout this paper, we denote by $\mathbb{R}$,$\mathbb{R}^+$ and $\mathbb{N}$ the set of all real numbers, the set of all non-negative real numbers and the set of all positive integers, respectively. Let $X$ be a non-empty set and $f$, $g$ be two self-mappings of $X$.
We denote by $F(f)$ the fixed point set of f,
that is., $F(f) = \{x \in X : fx = x\}$. Also, we denote by $F(f, g)$ the common fixed point set of $f$,$g$ that is., $F(f, g) = F(f) \cap F(g)$.

**Definition 1.** [7] Let $(X, \preceq, p)$ be an ordered partial metric space. we say that a mapping $f : X \to X$ is an almost generalized $C$-contractive mapping if there exist $\xi \geq 0$ and $(\psi, \phi) \in \Psi \times \Phi$ such that

$$\psi(p(fx, fy)) \leq \psi(M_p(x, y)) - \phi(M_p^{'}(x, y), M_p^{''}(x, y)) + \xi\psi(N_p(x, y)) - \phi(N_p^{'}(x, y), N_p^{''}(x, y)) \tag{1}$$

for all $x, y \in X$ with $x \preceq y$, where

$$\begin{aligned}
M_p(x, y) &= max\{p(x, y), p(x, fx), p(y, fy), \frac{p(x, fy) + p(y, fx)}{2}\},\\
M_p^{'}(x, y) &= max\{p(x, y), p(x, fx), p(x, fy)\},\\
M_p^{''}(x, y) &= max\{p(x, y), p(y, fy), p(y, fx)\},\\
N_p(x, y) &= min\{p(x, fx), p(y, fy), \frac{p(x, fy) + p(y, fx)}{2}\},\\
N_p^{'}(x, y) &= min\{p(x, fx), p(y, fx)\},\\
N_p^{''}(x, y) &= min\{p(y, fy), p(x, fy)\}.
\end{aligned}$$

**Definition 2.** [7] Let $(X, \preceq, p)$ be an ordered partial metric space, and let$f$,$g$ be two self mappings of $X$. The mapping $f$ is said to be almost generalized $C$-contractive with respect to $g$ if there exist $\xi \geq 0$ and $(\psi, \phi) \in \Psi \times \Phi$ such that

$$\psi(p(fx, gy)) \leq \psi(M_p(x, y)) - \phi(M_p^{'}(x, y), M_p^{''}(x, y)) + \xi\psi(N_p(x, y)) - \phi(N_p^{'}(x, y), N_p^{''}(x, y)) \tag{2}$$

for all $x, y \in X$ with $x \preceq y$, where

$$\begin{aligned}
M_p(x, y) &= max\{p(x, y), p(x, fx), p(y, gy), \frac{p(x, gy) + p(y, fx)}{2}\},\\
M_p^{'}(x, y) &= max\{p(x, y), p(x, fx), p(x, gy)\},\\
M_p^{''}(x, y) &= max\{p(x, y), p(y, gy), p(fx, y)\},\\
N_p(x, y) &= min\{p(x, fx), p(y, fy), \frac{p(x, gy) + p(y, fx)}{2}\},\\
N_p^{'}(x, y) &= min\{p(x, fx), p(y, fx), p(x, gy)\},\\
N_p^{''}(x, y) &= min\{p(y, fy), p(x, fy), p(y, gy)\}.
\end{aligned}$$

**lemma 3.** *[7] Let $(X, \preceq, p)$ be an ordered partial metric space. Assume that*

*$f : X \to X$ is an almost generalized $C$-contractive mapping. Fix $x_1 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If the sequence $\{x_n\}$ is non-decreasing and $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, then $\{x_n\}$ is a Cauchy sequence.*

**lemma 4.** *[7] Let $(X, \preceq, p)$ be an ordered partial metric space, and let $f$,$g$ be two self-mappings of $X$ which $f$ is an almost generalized $C$-contractive mapping with respect to $g$.Fix $x_1 \in X$ and define a sequence $\{x_n\}$ by $x_{2n} = fx_{2n-1}$ and $x_{2n+1} = gx_{2n}$ for all $n \in \mathbb{N}$. If the sequence $\{x_n\}$ is non-decreasing and $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, then $\{x_n\}$ is a Cauchy sequence.*

## 3 Main Results

In this section, we present some fixed point and common fixed point theorems for almost generalized $C$-contractive mappings in an ordered complete partial metric space.

**Theorem 5.** *Let $(X, \preceq, p)$ be an ordered complete partial metric space. let $f : X \to X$ be non-decreasing (with respect to $\preceq$), continuous and almost generalized $C$-contractive. If there exists $x_1 \in X$such that $x_1 \preceq fx_1$, then $f$ has a fixed point. In particular, if $F(f)$ is a totally ordered subset of $X$, then f has a unique fixed point.*

*Proof.* Define a sequence $\{x_n\}$ in $X$ by $x_1$ and $x_{n+1} = fx_n$ for all $n \in \mathbb{N}$. Since $x_1 \preceq fx_1 = x_2$ and $f$ is non-decreasing, we have $x_2 = fx_1 \preceq fx_2 = x_3$.By induction, One can show that
$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \ldots$.
If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = fx_{n_0}$, then $x_{n_0}$ is a fixed point of $f$. Hence the proof is complete. Now suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} M_p(x_{n-1}, x_n) &= max\{p(x_{n-1}, x_n), p(x_{n-1}, fx_{n-1}), p(x_n, fx_n), \frac{p(x_{n-1}, fx_n) + p(x_n, fx_{n-1})}{2}\}, \\ &= max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{p(x_{n-1}, x_{n+1}) + p(x_n, x_n)}{2}\}, \\ &\leq max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n) + p(x_n, x_n)}{2}\}, \\ &= max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}. \end{aligned} \tag{3}$$

$$\begin{aligned} M_p^{'}(x_{n-1}, x_n) &= max\{p(x_{n-1}, x_n), p(x_{n-1}, fx_{n-1}), p(x_{n-1}, fx_n)\}, \\ &= max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_{n-1}, x_{n+1})\}, \\ &\geq p(x_{n-1}, x_n). \end{aligned} \tag{4}$$

$$
\begin{aligned}
M_p^{''}(x_{n-1},x_n) &= max\{p(x_{n-1},x_n), p(x_n,fx_n), p(x_n,fx_{n-1})\},\\
&= max\{p(x_{n-1},x_n), p(x_n,x_{n+1}), p(x_n,x_n)\},\\
&\geq max\{p(x_{n-1},x_n), p(x_n,x_{n+1}), 0\},\\
&= max\{p(x_{n-1},x_n), p(x_n,x_{n+1})\}.
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
N_p(x_{n-1},x_n) &= min\{p(x_{n-1},fx_{n-1}), p(x_n,fx_n), \frac{p(x_{n-1},fx_n)+p(x_n,fx_{n-1})}{2}\},\\
&= min\{p(x_{n-1},x_n), p(x_n,x_{n+1}), \frac{p(x_{n-1},x_{n+1})+p(x_n,x_n)}{2}\},\\
&\geq 0.
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
N_p^{'}(x_{n-1},x_n) &= min\{p(x_{n-1},fx_{n-1}), p(x_n,fx_{n-1})\},\\
&= min\{p(x_{n-1},x_n), p(x_n,x_n)\},\\
&\geq 0.
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
N_p^{''}(x_{n-1},x_n) &= min\{p(x_n,fx_n), p(x_{n-1},fx_n)\},\\
&= min\{p(x_n,x_{n+1}), p(x_{n-1},x_{n+1})\},\\
&\geq 0.
\end{aligned} \tag{8}
$$

On the other hand, our hypothesis implies that there exists $\xi \geq 0$ and $(\psi, \phi) \in \Psi \times \Phi$ such that

$$\psi(p(fx,fy)) \leq \psi(M_p(x,y)) - \phi(M_p^{'}(x,y), M_p^{''}(x,y)) + \xi\psi(N_p(x,y)) - \phi(N_p^{'}(x,y), N_p^{''}(x,y))$$

for all $x, y \in X$ with $x \preceq y$,which yields

$$
\begin{aligned}
\psi(p(x_n,x_{n+1})) &= \psi(p(fx_{n-1},fx_n))\\
&\leq \psi(M_p(x_{n-1},x_n)) - \phi(M_p^{'}(x_{n-1},x_n), M_p^{''}(x_{n-1},x_n)) + \xi\psi(N_p(x_{n-1},x_n))\\
&\quad - \phi(N_p^{'}(x_{n-1},x_n), N_p^{''}(x_{n-1},x_n))
\end{aligned}
$$

for all $n \in \mathbb{N}$.since $N_p(x_{n-1},x_n) \geq 0$ and it takes only minimum value. So without loss of generality, we shall take $N_p(x_{n-1},x_n) = 0$, and equations (3)-(8) yield

$$
\begin{aligned}
\psi(p(x_n,x_{n+1})) &\leq \psi(max\{p(x_{n-1},x_n), p(x_n,x_{n+1})\})\\
&\quad - \phi(max\{p(x_{n-1},x_n), p(x_{n-1},x_{n+1})\}, max\{p(x_{n-1},x_n), p(x_n,x_{n+1})\}).
\end{aligned} \tag{9}
$$

holds for any $n \in \mathbb{N}$. Since $\phi$ and $\psi$ are non-decreasing. Thus, from (3),(4) and (9), we deduce that

$$
\begin{aligned}
\psi(p(x_n,x_{n+1})) &\leq \psi(max\{p(x_{n-1},x_n), p(x_n,x_{n+1})\})\\
&\quad - \phi(p(x_{n-1},x_n), max\{p(x_{n-1},x_n), p(x_n,x_{n+1})\}).
\end{aligned} \tag{10}
$$

holds for any $n \in \mathbb{N}$, which implies

$$\psi(p(x_n,x_{n+1})) < \psi(max\{p(x_{n-1},x_n), p(x_n,x_{n+1})\}) \tag{11}$$

holds for any $n \in \mathbb{N}$, because $p(x_n,x_{n+1}) > 0$, hence

$$\phi(p(x_{n-1},x_n), max\{p(x_{n-1},x_n), p(x_n,x_{n+1})\}) > 0$$

As $\psi$ is non-decreasing, from (11) it follows that

$$p(x_n, x_{n+1}) < max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}$$

holds for any $n \in \mathbb{N}$. This means that $p(x_n, x_{n+1}) < p(x_{n-1}, x_n)$ holds for all $n \in \mathbb{N}$. Thus the sequence $\{p(x_n, x_{n+1})\}$ is decreasing. Then it converges to some non-negative number $a$. Also from (10), for any $n \in \mathbb{N}$, we have

$$\psi(p(x_n, x_{n+1})) \leq \psi(p(x_{n-1}, x_n)) - \phi(p(x_{n-1}, x_n), p(x_{n-1}, x_n)).$$

The above inequality yields

$$\limsup_{n\to\infty} \psi(p(x_n, x_{n+1})) \leq \limsup_{n\to\infty} \psi(p(x_{n-1}, x_n)) - \liminf_{n\to\infty} \phi(p(x_{n-1}, x_n), p(x_{n-1}, x_n))$$

Consequently, we have

$$\psi(a) \leq \psi(a) - \phi(a, a),$$

which implies $\phi(a, a) = 0$. So $a = 0$. Then $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Now by lemma 2.1, the sequence $\{x_n\}$ is Cauchy. since $X$ is complete, there is some $z \in X$ such that $x_n \to z$ as $n \to \infty$. The continuity of $f$ implies $fx_n \to fz$ as $n \to \infty$.From the uniqueness of the limit, we conclude that $fz = z$. Hence $z \in F(f)$. Now, we suppose that $F(f)$ is totally ordered.
We will show that $z$ is unique.
Assume $u$ is another fixed point of $f$.
As $u, z \in F(f)$, our assumption implies that $z$ and $u$ are comparable.
Without loss of generality, we may assume that $u \preceq z$ and $N_p(u, z) = 0$.
Therefore,

$$\begin{aligned} \psi(p(u, z)) &= \psi(p(fu, fz)) \\ &\leq \psi(M_p(u, z)) - \phi(M_p^{'}(u, z), M_p^{''}(u, z)) + \xi\psi(N_p(u, z)) - \phi(N_p^{'}(u, z), N_p^{''}(u, z)). \\ &= \psi(p(u, z)) - \phi(p(u, z), p(u, z)). \end{aligned}$$

This yields $\phi(p(u, z), p(u, z)) = 0$. So $p(u, z) = 0$, that is, $u = z$.Therefore, $z$ is a unique fixed point. Thus, we get the desired result. □

The following corollary is an immediate consequence of the above theorem.

**Corollary 6.** *Let $(X, \preceq)$ and $(X, p)$ be a totally ordered set and a complete partial metric space, respectively. Let $f : X \to X$ be non-decreasing (with respect to $\preceq$), continuous and almost generalized $C$-contractive. If there exists $x_1 \in X$ such that $x_1 \preceq fx_1$,then $f$ has a unique fixed point.*

**Theorem 7.** *Let $(X, \preceq, p)$ be an ordered complete partial metric space, and let $f, g : X \to X$ be two weakly increasing mappings which $f$ is an almost generalized $C$-contractive mapping with respect to $g$. If either $f$ or $g$ is continuous, then the fixed point set of $f$ is non-empty and $F(f, g) = F(f) = F(g)$. Particularly, if $F(f)$ is a totally ordered subset of $X$, then $f$ and $g$ have a unique common fixed point.*

*Proof.* Our assumption implies that there exists some $(\psi, \phi, \xi) \in \Psi \times \Phi \times [0, \infty)$ such that

$$\psi(p(fx, gy)) \leq \psi(M_p(x, y)) - \phi(M_p^{'}(x, y), M_p^{''}(x, y)) + \xi\psi(N_p(x, y)) - \phi(N_p^{'}(x, y), N_p^{''}(x, y)) \tag{12}$$

for all $x, y \in X$ with $x \preceq y$, where

$$\begin{aligned}
M_p(x, y) &= max\{p(x, y), p(x, fx), p(y, gy), \frac{p(x, gy) + p(y, fx)}{2}\},\\
M_p^{'}(x, y) &= max\{p(x, y), p(x, fx), p(x, gy)\},\\
M_p^{''}(x, y) &= max\{p(x, y), p(y, gy), p(fx, y)\},\\
N_p(x, y) &= min\{p(x, fx), p(y, fy), \frac{p(x, gy) + p(y, fx)}{2}\},\\
N_p^{'}(x, y) &= min\{p(x, fx), p(y, fx), p(x, gy)\},\\
N_p^{''}(x, y) &= min\{p(y, fy), p(x, fy), p(y, gy)\}.
\end{aligned}$$

we now show that $F(f) = F(g)$. Let $z \in F(f)$. So $fz = z$. Since $z \preceq z$, inequality(12) implies that

$$\begin{aligned}
\psi(p(z, gz)) &= \psi(p(fz, gz))\\
&\leq \psi(M_p(z, z)) - \phi(M_p^{'}(z, z), M_p^{''}(z, z)) + \xi\psi(N_p(z, z)) - \phi(N_p^{'}(z, z), N_p^{''}(z, z))
\end{aligned}$$

Therefore,

$$\psi(p(z, gz)) \leq \psi(p(z, gz)) - \phi(p(z, gz), (p(z, gz)),$$

which yields $\phi(p(z, gz), (p(z, gz)) = 0$. As $\phi \in \Phi$, we get $p(z, gz) = 0$.
Hence $gz = z$, that is, $z \in F(g)$. So $F(f) \subseteq F(g)$.
Similarly, one can show that $F(g) \subseteq F(f)$.
Therefore, we have $F(f, g) = F(f) = F(g)$.
Let $x_1$ be an arbitrary element of $X$.
Define a sequence $\{x_n\}$ by $x_1$ and $x_{2n} = fx_{2n-1}$, $x_{2n+1} = gx_{2n}$ for all $n \in \mathbb{N}$
If there exists $m \in \mathbb{N}$ such that either $x_{2m} = x_{2m-1}$ or $x_{2m+1} = x_{2m}$ holds, then $F(f)$ is non-empty. Because if $x_{2m} = x_{2m-1}$, then $fx_{2m-1} = x_{2m} = x_{2m-1}$. So $x_{2m-1} \in F(f)$. If $x_{2m+1} = x_{2m}$, then $gx_{2m} = x_{2m+1} = x_{2m}$. Hence, $x_{2m} \in F(g) = F(f)$.
Therefore, we may suppose that $x_n \neq x_{n+1}$ for any $n \in \mathbb{N}$.
Without loss of generality, we can assume that $x_1 \preceq x_2$. We now show that the sequence $\{x_n\}$ is non-decreasing. As $f$ and $g$ are weakly increasing mappings, we obtain
$x_2 = fx_1 \preceq gfx_1 = gx_2 = x_3 \preceq fgx_2 = gx_3 = x_4 \preceq x_5 \preceq \ldots$. Hence the sequence $\{x_n\}$ is non-decreasing. Suppose $n \in \mathbb{N}$ is arbitrary. Since $x_{2n-1} \preceq x_{2n}$, inequality (12)implies

$$\begin{aligned}
\psi(p(x_{2n}, x_{2n+1})) &= \psi(p(fx_{2n-1}, gx_{2n}))\\
&\leq \psi(M_p(x_{2n-1}, x_{2n})) - \phi(M_p^{'}(x_{2n-1}, x_{2n}), M_p^{''}(x_{2n-1}, x_{2n})) \qquad (13)\\
&+ \xi\psi(N_p(x_{2n-1}, x_{2n})) - \phi(N_p^{'}(x_{2n-1}, x_{2n}), N_p^{''}(x_{2n-1}, x_{2n})).
\end{aligned}$$

where

$$M_p(x_{2n-1}, x_{2n}) = max\{p(x_{2n-1}, x_{2n}), p(x_{2n-1}, fx_{2n-1}), p(x_{2n}, gx_{2n}), \frac{p(x_{2n-1}, gx_{2n}) + p(x_{2n}, fx_{2n-1})}{2}\}$$
$$= max\{p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1}), \frac{p(x_{2n-1}, x_{2n+1}) + p(x_{2n}, x_{2n})}{2}\}$$
$$\leq max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1}), \frac{p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1})}{2}\}$$
$$= max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1})\}, \tag{14}$$

$$M_p^{'}(x_{2n-1}, x_{2n}) = max\{p(x_{2n-1}, x_{2n}), p(x_{2n-1}, fx_{2n-1}), p(x_{2n-1}, gx_{2n})\}$$
$$= max\{p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n+1})\} \tag{15}$$
$$\geq p(x_{2n-1}, x_{2n}),$$

$$M_p^{''}(x_{2n-1}, x_{2n}) = max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, gx_{2n}), p(fx_{2n-1}, x_{2n})\}$$
$$= max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1}), p(x_{2n}, x_{2n})\} \tag{16}$$
$$\geq p(x_{2n-1}, x_{2n})$$

$$N_p(x_{2n-1}, x_{2n}) = min\{p(x_{2n-1}, fx_{2n-1}), p(x_{2n}, fx_{2n}), \frac{p(x_{2n-1}, gx_{2n}) + p(x_{2n}, fx_{2n-1})}{2}\}$$
$$\geq 0, \tag{17}$$

Without loss of generality, we shall assume $N_p(x_{2n-1}, x_{2n}) = 0$

$$N_p^{'}(x_{2n-1}, x_{2n}) = min\{p(x_{2n-1}, fx_{2n-1}), p(x_{2n}, fx_{2n-1}), p(x_{2n-1}, gx_{2n})\}$$
$$= min\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n}), p(x_{2n-1}, x_{2n+1})\} \tag{18}$$
$$\geq min\{p(x_{2n-1}, x_{2n}), 0, p(x_{2n-1}, x_{2n+1})\}$$
$$= 0,$$

$$N_p^{''}(x_{2n-1}, x_{2n}) = min\{p(x_{2n}, fx_{2n}), p(x_{2n-1}, fx_{2n}), p(x_{2n}, gx_{2n})\}$$
$$= min\{p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n+1}), p(x_{2n}, x_{2n+1})\}$$
$$= min\{p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n+1})\} \tag{19}$$
$$\geq min\{p(x_{2n}, x_{2n}), p(x_{2n-1}, x_{2n-1})\}$$
$$\geq 0.$$

Thus, inequality (13) becomes,

$$\psi(p(x_{2n}, x_{2n+1})) \leq \psi(max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1}), \frac{p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1})}{2}\})$$
$$- \phi(max\{p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n+1})\}, max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1})\}). \tag{20}$$

Since $\psi$ and $\phi$ are non-decreasing, the above inequalities (14)to (19) yield the following inequality

$$\psi(p(x_{2n}, x_{2n+1})) \leq \psi(max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1})\})$$
$$- \phi(p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n})). \tag{21}$$

As $\phi(p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n})) > 0$, inequality (21) implies

$$\psi(p(x_{2n}, x_{2n+1})) < \psi(max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1})\})$$

Since $\psi$ is non-decreasing, it follows from the above inequality that

$$p(x_{2n}, x_{2n+1}) < max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1})\}$$

So

$$p(x_{2n}, x_{2n+1}) < max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1})\} = p(x_{2n-1}, x_{2n}) \qquad (22)$$

Hence inequality (21) becomes

$$\psi(p(x_{2n}, x_{2n+1})) \leq \psi(p(x_{2n-1}, x_{2n})) - \phi(p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n})). \qquad (23)$$

Similarly, one can show that

$$p(x_{2n+1}, x_{2n+2}) < p(x_{2n+1}, x_{2n}). \qquad (24)$$

Set $y_n = p(x_{2n}, x_{2n+1})$ and $p(x_{2n+1}, x_{2n+2})$. Then from (22) and (24), we get

$$\cdots < z_n < y_n < z_{n-1} < y_{n-1} < \cdots < z_1 < y_1, \qquad (25)$$

which shows that the two sequences $\{y_n\}$ and $\{z_n\}$ are strictly decreasing and bounded.
Hence $\{y_n\}$ and $\{z_n\}$ are convergent.
Assume that $\lim_{n\to\infty} y_n = a$ and $\lim_{n\to\infty} z_n = b$.
By (25), we have $a = b$. Taking the limit superior as $n \to \infty$ in (23), we conclude that

$$\limsup_{n\to\infty} \psi(p(x_{2n+1}, x_{2n})) \leq \limsup_{n\to\infty} \psi(p(x_{2n}, x_{2n-1})) - \liminf_{n\to\infty} \phi(p(x_{2n}, x_{2n-1}), p(x_{2n}, x_{2n-1})). \qquad (26)$$

Because $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = a$, (26), the continuity of $\psi$,and the lower semi-continuity of $\phi$ imply that

$$\psi(a) \leq \psi(a) - \phi(a, a).$$

Thus, $\phi(a, a) = 0$.Consequently, $a = 0$.
So $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0$.This implies that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.
As the sequence $\{x_n\}$ is non-decreasing and $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$,Lemma 2.2 implies that $\{x_n\}$ is a Cauchy sequence.
Since $X$ is complete, there is some $u \in X$ such that $x_n \to u$ as $n \to \infty$.
Without loss of generality, we assume that $f$ is continuous.
As $x_{2n-1} \to u$ as $n \to \infty$, the continuity of $f$ implies that $x_{2n} = fx_{2n-1} \to fu$ as $n \to \infty$.
By the uniqueness of the limit, we obtain $fu = u$. Therefore, $u \in F(f) = F(g)$.
Now suppose that $F(f)$ is a totally ordered subset of $X$.
We will show that $u$ is unique. Suppose that $z \in F(f, g) = F(f) = F(g)$.

By our hypothesis $u$, $z$ are comparable, hence without loss of generality, suppose $u \preceq z$.
Thus, inequality (12) implies that

$$\begin{aligned}\psi(p(u,z)) &= \psi(p(fu,gz)) \\ &\leq \psi(M_p(u,z)) - \phi(M_p^{'}(u,z), M_p^{''}(u,z)) + \xi\psi(N_p(u,z)) - \phi(N_p^{'}(u,z), N_p^{''}(u,z)) \\ &= \psi(p(u,z)) - \phi(p(u,z), p(u,z)).\end{aligned}$$

holds, which implies $\phi(p(u,z), p(u,z)) = 0$.
So $p(u,z) = 0$. Consequently, $u = z$.
This completes the proof of the theorem. □

Applying Theorem 3.2, we obtain the following result.

**Corollary 8.** *Let $(X, \preceq)$ and $(X, p)$ be a totally ordered set and a complete partial metric space, respectively. let $f, g : X \to X$ be two weakly increasing mappings which $f$ is an almost generalized $C$-contractive mapping with respect to $g$. If either $f$ or $g$ is continuous,then $f$ and $g$ have a unique common fixed point.*

The following example support Theorem 3.2.

**Example 9.** *Set $X = \{0,1,2,3\}$. define the metric $p$ on $X$ as follows:*

$$\begin{aligned}p(x,y) &= 0 \quad if x = y \\ &= x + y \quad if x \neq y\end{aligned}$$

*for all $x, y \in X$.*
*Define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by setting $\psi(t) = t^2$, $\phi(s,t) = \dfrac{s+t}{2}$ for all $s, t \in \mathbb{R}^+$, respectively.*
*Consider the relation $\preceq$ and the mappings $f$, $g$ on $X$ by $\preceq = \{(0,0),(1,1),(2,2),(3,3)\}$ and $f = \{(0,3),(1,1),(2,0),(3,3)\}$, $g = \{(0,1),(1,1),(2,3),(3,3)\}$, respectively.*
*It is clear that $(X, \preceq)$ is an ordered set.Then the following statements hold.*

*(i)* *$(X, \preceq, p)$ is an ordered complete partial metric space.*

*(ii)* *$f$ and $g$ are weakly increasing mappings with respect to $\preceq$.*

*(iii)* *$f$ is continuous.*

*(iv)* *$(\psi, \phi) \in \Psi \times \Phi$.*

*(v)* *$f$ is an almost generalized $C$-contractive mapping with respect to $g$.*

*We next show that*

$$\psi(p(fx,gy)) \leq \psi(M_p(x,y)) - \phi(M_p^{'}(x,y), M_p^{''}(x,y)) + \xi\psi(N_p(x,y)) - \phi(N_p^{'}(x,y), N_p^{''}(x,y)) \quad (27)$$

*for all* $x, y \in X$ *with* $x \preceq y$*, where* $\xi \geq 10.5$*, and*

$$\begin{aligned}
M_p(x,y) &= max\{p(x,y), p(x,fx), p(y,gy), \frac{p(x,gy)+p(y,fx)}{2}\},\\
M_p^{'}(x,y) &= max\{p(x,y), p(x,fx), p(x,gy)\},\\
M_p^{''}(x,y) &= max\{p(x,y), p(y,gy), p(fx,y)\},\\
N_p(x,y) &= min\{p(x,fx), p(y,fy), \frac{p(x,gy)+p(y,fx)}{2}\},\\
N_p^{'}(x,y) &= min\{p(x,fx), p(y,fx), p(x,gy)\},\\
N_p^{''}(x,y) &= min\{p(y,fy), p(x,fy), p(y,gy)\}.
\end{aligned}$$

*To see this, we have*

$$\begin{aligned}
16 &= \psi(p(f(0),g(0)))\\
&\leq \psi(M_p(0,0)) - \phi(M_p^{'}(0,0), M_p^{''}(0,0)) + \xi\psi(N_p(0,0)) - \phi(N_p^{'}(0,0), N_p^{''}(0,0))\\
&= \psi(3) - \phi(3,3) + \xi\psi(2) - \phi(1,1)\\
&= 5 + 4\xi
\end{aligned}$$

$$\begin{aligned}
0 &= \psi(p(f(1),g(1)))\\
&\leq \psi(M_p(1,1)) - \phi(M_p^{'}(1,1), M_p^{''}(1,1)) + \xi\psi(N_p(1,1)) - \phi(N_p^{'}(1,1), N_p^{''}(1,1))\\
&= \psi(0) - \phi(0,0) + \xi\psi(0) - \phi(0,0)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
9 &= \psi(p(f(2),g(2)))\\
&\leq \psi(M_p(2,2)) - \phi(M_p^{'}(2,2), M_p^{''}(2,2)) + \xi\psi(N_p(2,2)) - \phi(N_p^{'}(2,2), N_p^{''}(2,2))\\
&= \psi(5) - \phi(5,5) + \xi\psi(2) - \phi(2,2)\\
&= 18 + 4\xi
\end{aligned}$$

$$\begin{aligned}
0 &= \psi(p(f(3),g(3)))\\
&\leq \psi(M_p(3,3)) - \phi(M_p^{'}(3,3), M_p^{''}(3,3)) + \xi\psi(N_p(3,3)) - \phi(N_p^{'}(3,3), N_p^{''}(3,3))\\
&= \psi(0) - \phi(0,0) + \xi\psi(0) - \phi(0,0)\\
&= 0
\end{aligned}$$

*These mean that the mapping* $f$ *is almost generalized* $C$*-contractive with respect to the mapping* $g$*. Now, it follows from Theorem 3.2 that the fixed point set of* $f$ *is non-empty and* $F(f,g) = F(f) = F(g)$*. We observe that* $F(f,g) = F(f) = F(g) = \{1,3\}$

## References

[1] Ciric, LB, Abbas,M, Saadati,R, Hussain, N:Common fixed points of almost generalized contractive mappings in ordered metric spaces,*Appl. Math.Comput.***217**, (2011), 5784-5789.

[2] Shatanawi,W, Al-Rawashdeh, *A Common fixed point of almost generalized* $(\psi, \phi)$*-contractive mappings in ordered metric spaces. Fixed Point Theory Appl*(80)(2012).

[3] Banayak S.Choudhury and Amaresh kundu: *Weak contraction in partial metric spaces.*(1994).

[4] Daniela Paesano and Pasquale Vetro: *Common fixed points in a partially ordered partial metric space.*

[5] S.G.Mathews: Partial mteric topology,*Annals of Newyork Academy of sciences* **728**,183-197(1994).

[6] Azizollah Azizi, Mohammad Moosaei and Gita Zarei*Fixed point for almost generalized* $C$*-contractive mappings in ordered complete metric spaces.Fixed Point theory and Appl*(80) (2016).

[7] A.Jennie Sebasty Pritha and Dr.U.Karuppiah: Fixed Point theorems for almost generalized $C$-contractive mappings in ordered partial metric spaces,*Journal of Global Research Mathematical Archives-Communicated.*