New Family of Heinz Type Means

R. Sampath Kumar, G. Narasimhan and K.M. Nagaraja

1Research & Development Centre,
Bharathiar University,
Coimbatore, Tamilnadu.
Department of Mathematics,
R N S Institute of Technology,
Uttarahalli-Kengeri Main Road,
R R Nagar Post, Bengaluru.
r.sampathkumar1967@gmail.com

2Department of Mathematics,
R N S Institute of Technology,
Uttarahalli-Kengeri Main Road,
R R Nagar Post, Bengaluru.
narasimhan.guru@gmail.com

3Department of Mathematics,
J.S.S. Academy of Technical Education,
Uttarahalli-Kengeri Main Road,
Bengaluru, Karnataka India.
nagkmn@gmail.com

Abstract

In this paper, the new family of means of Heinz type means are introduced, further the basic properties, monotonic results, Log-Convexity and Concavity results and some deductions are established.
1. Introduction

For two positive real numbers \(a\) & \(b\) Arithmetic Mean: \(A(a, b) = \frac{(a + b)}{2}\),
Geometric Mean: \(G(a, b) = \sqrt{ab}\), Harmonic Mean: \(H(a, b) = \frac{2ab}{a + b}\), Contra Harmonic Mean:
\(C(a, b) = \frac{a^2 + b^2}{a + b}\), and Power Mean
\(M_r(a, b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}}\) \((r \neq 0)\).

For \(0 \leq v \leq 1\), \(a, b > 0\), the family of Heinz means are defined as,
\(H_v(a, b) = \frac{a^{1-v}b^v + a^v b^{1-v}}{2}\).

For \(v = 0, 1\); \(H_v(a, b) = A(a, b)\) and for \(v = \frac{1}{2}\); \(H_v(a, b) = A(a, b)\) That is \(H_v(a, b)\) as a function \(v\) is convex and attains minimum value at \(v = \frac{1}{2}\) and proved that \(G(a, b) \leq H_v(a, b) \leq A(a, b)\) for \(0 \leq v \leq 1\).

The interesting results on family of Heron means \(H_a(a, b) = (1 - a) G(a, b) + a A(a, b)\) and Heinz mean were found in [2]. In [3, 4] authors studied the family of heron means in different notations and established some inequalities with power mean, Identric mean and Logarithmic mean.

For \(0 \leq v \leq 1\), let \(\alpha(v) = 1 - 4(v - v^2)\), the function \(\alpha(v)\) is convex in nature and attains minimum value at \(v = \frac{1}{2}\) and maximum value at \(v = 0, 1\).

In [6], K M Nagaraja and et.,al, proved that \(H(a, b) + C(a, b) = 2A(a, b)\) and also established some double inequalities involving means.

Several authors introduced and studied in depth the parameterized family of means such as Stolorsky’s mean; functional means; Heinz means; etc., found generous and interesting results in ([1],[5],[8],[12]-[15],[17]-[20]).

In [7,9,16] authors studied some results related to Centroidal mean which is defined by \(C(a, b) = \frac{2}{3} \left(\frac{a^2 + ab + b^2}{a + b}\right)\) which is equivalently written as \(CT(a, b) = \frac{2}{3} \left(\frac{a^2 + ab + b^2}{a + b}\right)\). This work further continued to introduce the mean similar to Heinz family of means and establish the improvement in Harmonic and Contra Harmonic mean inequality chain. This work motivates us to develop this paper.

2. Definition and Results

In this section, the new family of Heinz type means is defined as follows;

**Definition 2.1.** [4,19] For \(a, b > 0\), a mean \(M(a, b)\) of \(a\) and \(b\) is defined as the function \(M(a, b): R^2_+ \rightarrow R_+\), which has the property that
\[a \land b \leq M(a, b) \leq a \lor b\] \((2.1)\)
where \( a\wedge b = \text{Min}(a, b) \) and \( a\vee b = \text{Max}(a, b) \).

**Definition 2.2.** For \( 0 \leq v \leq 1, \ a, b > 0 \), \( H_v^*(a, b) = \frac{a^{1+v}b^{1-v}+a^{1-v}b^{1+v}}{a^{1+v}b^{1-v}+a^{1-v}b^{1+v}} \) are called the new family of Heinz type means.

The new family of Heinz type means, \( H_v^*(a, b) \) for two distinct positive real values \( a \) & \( b \) and the parameter \( v \) belongs to \([0, 1]\). The parameter \( v \) can be extending either side of the interval \([0, 1]\). Firstly, verify that \( H_v^*(a, b) \) satisfies the properties and definitions of mean.

**Lemma 2.1.** The new family of Heinz type means \( H_v^*(a, b) \) for two distinct positive real values \( a \) & \( b \) is a mean for all values of \( v \).

**Proof.** According to the definition of mean we need to verify the condition \( \text{Min}(a, b) \leq H_v^*(a, b) \leq \text{Max}(a, b) \) for all values of \( v \) and \( a < b \) this is achieved by considering the following two cases;

Case(i) consider \( a - H_v^*(a, b) = a - \frac{a^{1+v}b^{1-v}+a^{1-v}b^{1+v}}{a^{1+v}b^{1-v}+a^{1-v}b^{1+v}} \), since \( a < b \) and on simplification gives \( a - H_v^*(a, b) = \frac{a^{1-v}b^{1+v}+(a-b)}{a^{1+v}b^{1-v}+a^{1-v}b^{1+v}} \leq 0 \)

This proves that \( a - H_v^*(a, b) < 0 \).

Case(ii) consider \( b - H_v^*(a, b) = b - \frac{a^{1+v}b^{1-v}+a^{1-v}b^{1+v}}{a^{1+v}b^{1-v}+a^{1-v}b^{1+v}} \), since \( a < b \) and on simplification gives

\[
b - H_v^*(a, b) = \frac{a^{1-v}b^{1+v}(b-a)}{a^{1+v}b^{1-v}+a^{1-v}b^{1+v}} \geq 0.
\]

This proves that \( b - H_v^*(a, b) > 0 \).

Hence the proof of lemma 2.1.

**Proposition 2.1.** The new family of Heinz type means \( H_v^*(a, b) \) are symmetric and homogeneous;

- **Symmetric:** \( H_v^*(a, b) = H_v^*(b, a) \)
- **Homogeneous:** \( H_v^*(at, bt) = t H_v^*(a, b) \).

**Theorem 2.1.** The new family of Heinz type means \( H_v^*(a, b) \) for two distinct positive real values \( a \) & \( b \) is monotonically increasing function with respect to the parameter \( v \).

**Proof.** Consider

\[
H_{v+1}^*(a, b) - H_v^*(a, b) = \frac{a^{2+v}b^{2-v}+a^{2-v}b^{2+v}}{a^{2+v}b^{2-v}+a^{2-v}b^{2+v}} - \frac{a^{1+v}b^{1-v}+a^{1-v}b^{1+v}}{a^{1+v}b^{1-v}+a^{1-v}b^{1+v}}
\]

on simplification gives

\[
H_{v+1}^*(a, b) - H_v^*(a, b) = \frac{(a-b)^2(a+b)}{(a^{1+v}b^{1-v}+a^{1-v}b^{1+v})(a^{1+v}b^{1-v}+a^{1-v}b^{1+v})} \geq 0.
\]

This proves that \( H_{v+1}^*(a, b) > H_v^*(a, b) \). Hence the proof of the theorem 2.1.
3. Some Deductions

In this section some well known means are extracted from the new family of Heinz type means $H_v^*(a, b)$.

**Proposition 3.1.** For two distinct positive real values $a \& b$ and for some particular values of $v$, the new family of Heinz type means $H_v^*(a, b)$ becomes.

- For $v = 0$, $H_0^*(a, b) = H(a, b)$, the Harmonic mean
- For $v = \frac{1}{2}$, $H_{1/2}^*(a, b) = A(a, b)$, the Arithmetic mean
- For $v = 1$, $H_1^*(a, b) = C(a, b)$, the Contra Harmonic mean
- For $v = 2$, $H_2^*(a, b) = C_3(a, b)$, the generalized Contra Harmonic mean

for $n = 3$

where $C_n(a, b) = \frac{a^n + b^n}{a^{n+1}b^{-n}}$ is the generalized contra-harmonic mean and is also called Lehmer mean.

**Remark 3.1.** For $v = -1$, $H_{-1}^*(a, b) = \frac{ab(a^2 + b^2)}{a^2 + b^2}$, which is equivalently written as $G^2(a, b) = H_{-1}^*(a, b)C_2(a, b) = A(a, b)H(a, b)$ which is an identity.

4. Log-Convexity and Concavity

In [10, 11], authors discussed some results on Log-Convexity and Concavity of some Double Sequences and the Log-convexity of generalized Contra Harmonic mean $C_n(a, b) = \frac{a^n + b^n}{a^{n+1}b^{-n}}$ is also called Lehmer mean.

**Definition 4.1.** The sequence $C_n(a, b) = \frac{a^n + b^n}{a^{n+1}b^{-n}}$ is said to be log-convex iff $c_n^2 \leq c_{n+1}c_{n-1}$ and log-concave iff $c_n^2 \geq c_{n+1}c_{n-1}$.

**Theorem 4.1.** The new family of Heinz type means $H_v^*(a, b)$ for two distinct positive real values $a \& b$ is log-convex with respect to the parameter $v$ if (i) $a > b$ and $v \leq 1/4$ (ii) $a < b$ and $v \geq 1/4$ otherwise log-concave.

**Proof.** Consider,

$$
H_{v+1}^*H_{v-1}^* - (H_v^*)^2 = \left(\frac{a^{2+v}b^{-v}+a^{-v}b^{2+v}}{a^{1+v}b^{-v}+a^{-v}b^{1+v}}\right)^2
- \left(\frac{a^{1+v}b^{1-v}+a^{-1-v}b^{1+v}}{a^v b^{1-v} + a^{-v} b^v}\right)^2
$$

on simplification gives

$$
H_{v+1}^*H_{v-1}^* - (H_v^*)^2 = \left(\frac{a^{3+2v}b^{3-2v} + a^5 b + a b^5 + a^3 - 2v b^3 + 2v}{a^{1+2v}b^{3-2v} + a^4 + b^4 + a^3 - 2v b^1 + 2v}\right)
- \left(\frac{a^{2+2v}b^{2-2v} + 2a^2 b^2 + a^2 - 2v b^2 + 2v}{a^{2v} b^{2-2v} + 2a b + a^2 - 2v b^2}\right)
$$

regrouping the terms leads to $H_{v+1}^*H_{v-1}^* - (H_v^*)^2 = (a - b)\Delta$, where
\[ 
\Delta = \frac{b^4(a^2 - 2v b^{1+2v} - a^{1+2v} b^{1-2v}) + a^4(a^{2-2v} b^{1+2v} - a^{1+2v} b^{2-2v}) - 2a^2 b^2(a^2 - 2v b^{1+2v} - a^{1+2v} b^{2-2v})}{(a^{1+2v} b^{1-2v} + a^4 + b^4 + a^{3-2v} b^{1+2v}) (a^{1+2v} b^{1+2v} + a^4 + b^4 + a^{3-2v} b^{1+2v})} 
\]

which is equivalently written as

\[ 
H_{v+1}^* H_{v-1}^* - (H_v^*)^2 = \left\{ \begin{array}{ll} 
(a - b) + (a^2 - b^2)^2 (a^{2-2v} b^{1+2v} - a^{1+2v} b^{2-2v}) 
& 
(a^{1+2v} b^{3-2v} + a^4 + b^4 + a^{3-2v} b^{1+2v}) (a^{1+2v} b^{3-2v} + a^4 + b^4 + a^{3-2v} b^{1+2v}) 
\end{array} \right. 
\]

from above expression we have the following conclusions.

- For \( a > b \), then \( (a^{2-2v} b^{1+2v} - a^{1+2v} b^{2-2v}) > 0 \) if \( v \leq 1/4 \)
- For \( a < b \), then \( (a^{2-2v} b^{1+2v} - a^{1+2v} b^{2-2v}) < 0 \) if \( v \geq 1/4 \)

This proves that \( H_{v+1}^* H_{v-1}^* - (H_v^*)^2 > 0 \), that is \( H_v^* (a, b) \) is log convex and otherwise \( H_v^* (a, b) \) is log-concave. Hence the proof of the theorem (4.1).

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**References**


