

On the results of Hilfer fractional derivative with nonlocal conditions

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Abstract

This paper concerned with the existence of nonlocal initial value problem with Hilfer fractional derivative. The results are obtained by using suitable fixed point theorem combined with fractional calculus theory.

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1 Introduction

Fractional differential equations have gained importance and popularity during past few decades, mainly due to its demonstrative applications in numerous. Seemingly diverse fields of science and engineering. For example, the nonlinear oscillation of earth quake can be modulated with fractional derivatives, and the Fluid dynamic traffic model with fractional derivatives can eliminate the differences arising from the assumption of continuum traffic flow. Due to tremendous scope and applications, several have been devoted to study the existence of mild solution of fractional integro-differential equations studied by many authors [1–8, 13–21, 29–31].

Two parameter family of fractional derivatives $D_{a+}^{\alpha,\beta}$ of order α and β allows interpolate between the Riemann-Liouville and Caputo's derivatives described in [10, 12] and the references therein. Hilfer et. al [11, 27] have initially proposed linear differential with new fractional operator. Hilfer fractional derivatives and applied operational calculus used to solve such simple fractional integro-differential equations studied by many authors in [9, 12, 23–26, 28].

Stimulated by the above discussion, In this paper we extend the study of existence of solutions to nonlocal initial value problem for Hilfer type integro- differential equations of

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the form

$$D_{a+}^{\alpha,\beta} x(t) = f(t, x(t), T(x(t)), S(x(t))), \quad 0 < \alpha < 1, 0 \leq \beta \leq 1, t \in (a, b] \quad (1)$$

$$I_{a+}^{1-\gamma} x(a^+) = \sum_{i=1}^m \lambda_i x(\tau_i), \quad \alpha \leq \gamma = \alpha + \beta - \alpha\beta < 1, \tau_i \in (a, b] \quad (2)$$

where $T(x(t)) = \int_0^t K(t, s)x(s)ds$ and $S(x(t)) = \int_0^t H(t, s)x(s)ds$.

The operator $D_{a+}^{\alpha,\beta}$ is generalized Riemann-Liouville and Caputo', fractional derivative operator. The operator $I_{a+}^{1-\gamma}$ denotes the left-sided Riemann-Liouville fractional integral as given in section 2. The nonlinear forms $f : (a, b] \times R^3 \rightarrow R^3$ is a given function, $\tau_i, i = 1, 2, 3, \dots, m$ are prefixed point satisfying $a \leq \tau_1 \leq \tau_2 \leq \dots \leq b$.

Also we implement some ideas in Eq.1 to establish an equivalent mixed type integral equation

$$\begin{aligned} x(t) = & \frac{T(t-a)^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds. \end{aligned} \quad (3)$$

where

$$T = \frac{1}{\Gamma(\gamma) - \sum_{i=1}^m \lambda_i (\tau_i - a)^{\gamma-1}} \text{ in the weighted space of continuous functioning } C_{1-\gamma}[a, b]$$

using different fixed point theorem such as Krasnoselskii fixed point theorem, Schauder fixed point theorem and Schaefer fixed point theorem.

This paper is arranged as follows. Initially in Section 2, we establish a set of sufficient conditions for the existence of fractional integro-differential equations in Banach spaces. In Section 3, existence of solutions has been proved using some fixed point theorems. Finally in Section 4, an example is given to illustrate the achieved results.

2 Preliminaries

Initially we recall some basic definitions of the Riemann - Liouville fractional integral and derivative which will be made up to the Hilfer fractional derivative.

Let us recall the following known definitions. For more details see [27, Chapter 1].

Definition 2.1. (see [13]) The left-sided Riemann-Liouville fractional derivative of order $\alpha \in R^+$ of function f is defined as

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(s)ds}{(x-s)^{1-\alpha}}, \quad x > 0, \quad \alpha > 0,$$

where $\Gamma(\cdot)$ is the gamma function,

Definition 2.2. (see [13]) The left-sided Riemann-Liouville fractional derivative of order $\alpha \in [n-1, n)$, $n \in Z^+$, of function f is defined as

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_0^x \frac{f(s)ds}{(x-s)^{\alpha-n+1}}, \quad x > 0, \quad \alpha > 0,$$

where $\Gamma(\cdot)$ is the gamma function,

Definition 2.3. (see [11]) The left-sided Hilfer fractional derivative of order $0 < \alpha < 1$ and $0 \leq \beta \leq 1$ of function f is defined as

$$D_{a+}^{\alpha,\beta} f(x) = \left(I_{a+}^{\beta(1-\alpha)} D \left(I_{a+}^{(1-\beta)(1-\alpha)} f \right) \right)(x).$$

where $D = \frac{d}{dx}$.

The Hilfer fractional derivative is considered as an interpolator between the Riemann-Liouville and Caputo derivative, then the remark provides the relation with mentioned above

Remark 2.4. (see [11])

- (i) The operator $D_{a+}^{\alpha,\beta}$ also can be rewritten as

$$D_{a+}^{\alpha,\beta} = I_{a+}^{\beta(1-\alpha)} D I_{a+}^{(1-\beta)(1-\alpha)} = I_{a+}^{\beta(1-\alpha)} D_{a+}^{\gamma}, \quad \gamma = \alpha + \beta - \alpha\beta.$$
- (ii) The left-sided Riemann-Liouville fractional derivative can be represented as $D_{a+}^{\alpha} = D_{a+}^{\alpha,0}$ if $\beta = 0$
- (iii) The left-sided Caputo fractional derivative can be represented as ${}^c D_{a+}^{\alpha} = I_{a+}^{1-\alpha} D$ if $\beta = 1$

Next, we need the following basic work spaces.

Let $0 < a < b < \infty$ and $C[a, b]$, the Banach space of all continuous functions from $[a, b]$ into $\mathbb{R}$ with the norm $\|x\|_c = \max\{|x(s)| : s \in [a, b]\}$. For $0 \leq \gamma < 1$, we denote the space $C_{\gamma}[a, b]$ as

$$C_{\gamma}[a, b] = \{f(x) : (a, b) \rightarrow \mathbb{R} | (x-a)^{\gamma} f(x) \in C[a, b]\}.$$

where $C_{\gamma}[a, b]$ is the weighted space of the continuous functions f in the finite interval $[a, b]$ with the norm

$$\|f\|_{C_{\gamma}} = \|(x-a)^{\gamma} f(x)\|_C.$$

Meanwhile, $C_{\gamma}^n = \{f \in C^{n-1}[a, b]\}$ is the Banach space with the norm

$$\|f\|_{C_{\gamma}^n} = \sum_{i=0}^{n-1} \|f^{(i)}\|_C + \|f^{(n)}\|_{C_{\gamma}}, \quad n \in \mathbb{N}.$$

Moreover, $C_{\gamma}^0[a, b] = C_{\gamma}[a, b]$.

Now, we review the following lemmas which is useful in the following sequels.

Lemma 2.5. (see [13], p.74) If $\alpha > 0$ and $\beta > 0$, there exists

$$\left(I_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1},$$

and

$$\left(D_{a+}^{\alpha}(t-a)^{\alpha-1}\right)(x) = 0, \quad 0 < \alpha < 1.$$

In particular, if $f \in C_{\gamma}[a, b]$ or $c[a, b]$, the

Lemma 2.6. (see [13], lemma 2.5) The following properties exists if $\alpha > 0$, $\beta > 0$ and $f \in L^1(a, b)$, for $x \in [a, b]$

$$\left(I_{a+}^{\alpha} I_{a+}^{\beta} f\right)(x) = \left(I_{a+}^{\alpha+\beta} f\right)(x) \quad \text{and} \quad \left(D_{a+}^{\alpha} I_{a+}^{\alpha} f\right)(x) = f(x).$$

In particular, if $f \in C_{\gamma}[a, b]$ or $f \in C[a, b]$, then these equalities hold at each $x \in (a, b]$ or $x \in [a, b]$, respectively.

Lemma 2.7. (see [7]) If $f \in C_{\gamma}[a, b]$ and $I_{a+}^{1-\alpha} f \in C_{\gamma}^1[a, b]$ for $0 < \alpha < 1$, $0 \leq \gamma < 1$, then

$$I_{a+}^{\alpha} D_{a+}^{\alpha} f(x) = f(x) - \frac{I_{a+}^{1-\alpha} f(a)}{\Gamma(a)}(x-a)^{\alpha-1}, \quad \text{for all } x \in (a, b].$$

Lemma 2.8. (see [7]) For $0 \leq \gamma < 1$ and $f \in C_\gamma[a, b]$, then

$$I_{a+}^\alpha f(a) = \lim_{x \rightarrow a+} I_{a+}^\alpha f(x) = 0, \quad 0 \leq \gamma < \alpha.$$

In order to solve our problem, the following spaces are presented

$$C_{1-\gamma}^{\alpha, \beta} = \{f \in C_{1-\gamma}[a, b], D_{a+}^{\alpha, \beta} f \in C_{1-\gamma}[a, b]\}.$$

and

$$C_{1-\gamma}^\gamma = \{f \in C_{1-\gamma}[a, b], D_{a+}^\gamma f \in C_{1-\gamma}[a, b]\}.$$

It is true that

$$C_{1-\gamma}^\gamma[a, b] \subset C_{1-\gamma}^{\alpha, \beta}[a, b].$$

Lemma 2.9. (see [7], lemma 20) Let $\alpha > 0$, $\beta > 0$ and $\gamma = \alpha + \beta - \alpha\beta$. If $f \in C_{1-\gamma}^\gamma[a, b]$, then

$$I_{a+}^\gamma D_{a+}^\gamma f = I_{a+}^\alpha D_{a+}^{\alpha, \beta} f, \quad D_{a+}^\gamma I_{a+}^\alpha f = D_{a+}^{\beta(1-\alpha)} f.$$

Lemma 2.10. (see [7]) Let $f \in L^1[a, b]$ and $D_{a+}^{\beta(1-\alpha)} f \in L^1[a, b]$, then

$$D_{a+}^{\alpha, \beta} I_{a+}^\alpha f = I_{a+}^{\beta(1-\alpha)} D_{a+}^{\beta(1-\alpha)} f.$$

Lemma 2.11. (see [7]) Let $f : (a, b) \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma}[a, b]$ for any $x \in C_{1-\gamma}[a, b]$. A function $x \in C_{1-\gamma}[a, b]$ is a solution of fractional value problem:

$$\begin{cases} D_{a+}^{\alpha, \beta} x(t) = f(t, x(t), T(x(t))), & 0 < \alpha < 1, \quad 0 \leq \beta \leq 1, \\ I_{a+}^\gamma x(a^+) = x_a, & \gamma = \alpha + \beta - \alpha\beta. \end{cases}$$

if and only if x satisfies the following Volterra integral equation,

$$x(t) = \frac{x_a(t-a)^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds.$$

By the above results the new mixed type Volterra integral for our problem will be given in the next lemma.

Lemma 2.12. Let $f : (a, b) \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma}[a, b]$ for any $x \in C_{1-\gamma}[a, b]$. A function $x \in C_{1-\gamma}[a, b]$ is a solution of [1] if and only if x satisfies the mixed type integral [3].

Proof. According to lemma ??, a solution of 1 can be expressed as

$$x(t) = \frac{I_{a+}^{1-\gamma} x(a^+)}{\Gamma(\gamma)} (t-a)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds. \quad (4)$$

Now substitute $t = \tau_i$ in the above equation

$$x(\tau_i) = \frac{I_{a+}^{1-\gamma} x(a^+)}{\Gamma(\gamma)} (\tau_i - a)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds. \quad (5)$$

Multiplying λ_i on both sides of [5], becomes

$$\lambda_i x(\tau_i) = \frac{I_{a+}^{1-\gamma} x(a^+)}{\Gamma(\gamma)} \lambda_i (\tau_i - a)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds. \quad (6)$$

Then we have

$$\begin{aligned} I_{a^+}^{1-\gamma} x(a^+) &= \sum_{i=1}^m \lambda_i x(\tau_i) \\ &= \frac{I_{a^+}^{1-\gamma} x(a^+)}{\Gamma(\gamma)} \sum_{i=1}^m \lambda_i (\tau_i - a)^{\gamma-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds. \end{aligned}$$

which implies

$$I_{a^+}^{1-\gamma} x(a^+) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} T \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds. \quad (7)$$

Substituting [7] in [4] we get [3]. It is proved that x is also a solution of eq[1]. Next we prove the sufficiency by applying $I_{a^+}^{1-\gamma}$ to both sides of the integral Eq.[3], we have

$$\begin{aligned} I_{a^+}^{1-\gamma} x(t) &= I_{a^+}^{1-\gamma} (t-a)^{\gamma-1} \frac{T}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds \\ &\quad + I_{a^+}^{1-\gamma} I_{a^+}^{\alpha} f(t, x(t), T(x(t)), S(x(t))), \end{aligned}$$

using lemmas 2.5 and 2.6,

$$\begin{aligned} I_{a^+}^{1-\gamma} x(t) &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} T \sum_{i=1}^m \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds \\ &\quad + I_{a^+}^{1-\beta(1-\alpha)} f(t, x(t), T(x(t)), S(x(t))). \end{aligned} \quad (8)$$

Since $1-\gamma < 1-\beta(1-\alpha)$. Lemma 2.8 can be used when taking the limit as $t \rightarrow a$.

$$I_{a^+}^{1-\gamma} x(a) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} T \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds.$$

Put $t = \tau_i$ into eq.3, we have

$$\begin{aligned} x(\tau_i) &= \frac{T}{\Gamma(\alpha)} (\tau_i - a)^{\gamma-1} \sum_{i=1}^m \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{i=1}^m \lambda_i x \tau_i &= \frac{T}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds \sum_{i=1}^m \lambda_i (\tau_i - a)^{\gamma-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds \\ &= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds \left(1 + T \sum_{i=1}^m \lambda_i (\tau_i - a)^{\gamma-1} \right) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} T \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds. \end{aligned}$$

It follows (8) and (9) that

$$I_{a+}^{1-\gamma} x(a) = \sum_{i=1}^m \lambda_i x(\tau_i).$$

Now by applying D_{a+}^{γ} to both sides of 2, it follows from Lemmas 2.5 and 2.9 that

$$D_{a+}^{\gamma} u(t) = D_{a+}^{\beta(1-\alpha)} f(t, x(t), T(x(t)), S(s(t))). \quad (9)$$

Since $x \in C_{1-\gamma}^{\gamma}[a, b]$ and by definition of $C_{1-\gamma}^{\gamma}[a, b]$, we have $D_{a+}^{\gamma} x \in C_{1-\gamma}^{\gamma}[a, b]$, then $D_{a+}^{\beta(1-\alpha)} f = DI_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma}[a, b]$. For $f \in C_{1-\gamma}[a, b]$, it is obvious that $I_{a+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma}[a, b]$. Thus f and $I_{a+}^{1-\beta(1-\alpha)} f$ satisfy the conditions of Lemma 2.7.

Next, applying $I_{a+}^{\beta(1-\alpha)}$ both sides of Eq.(2) and using Lemma 2.7,

$$D_{a+}^{\alpha, \beta} x(t) = f(t, x(t), T((t)), S(x(t))) - \frac{I_{a+}^{1-\beta(1-\alpha)} f(a)}{\Gamma(\beta(1-\alpha))} (t-a)^{\beta(1-\alpha)-1},$$

where $I_{a+}^{1-\beta(1-\alpha)} f(a) = 0$ is implied by Lemma 2.8. Hence it reduces to

$$D_{a+}^{\alpha, \beta} x(t) = f(t, x(t), T((t)), S(x(t))).$$

The results proved completely. $\square$

3 Existence results

The following assumptions are introduced in this section.

(H₁): $f : (a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function such that $f(\cdot, x(\cdot), T(x(\cdot)), S(x(\cdot))) \in C_{1-\gamma}^{\beta(1-\alpha)}[a, b]$ for any $x \in C_{1-\gamma}^{\gamma}[a, b]$. For all $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$, there exists a positive constant $L_1, L_2, L_3 > 0$ such that

$$|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)| \leq L_1 \|x_1 - y_1\| + L_2 \|x_2 - y_2\| + L_3 \|x_3 - y_3\|,$$

where $t \in (a, b]$, $x, y \in X$

(H₂): Denote $K^* = \sup_{t \in (a, b]} \int_0^t |k(s, t)| dt < \infty$, $H^* = \sup_{t \in (a, b]} \int_0^t |H(s, t)| dt < \infty$ and $L = \max\{L_1, L_2, L_3\}$.

(H₃): The constant

$$\varrho = \frac{LB(\gamma, \alpha)}{\Gamma(\alpha)} \left(|T| \sum_{i=1}^m \lambda_i (\tau_i - a)^{\alpha+\gamma-1} + (b-a)^{\alpha} \right) < 1.$$

(H₄): There exists $L, L_1, L_2, L_3, K^*, H^*$ satisfy the inequality

$$\left[LB(\gamma, \alpha)(1 + K^* + H^*) \right] < 1.$$

Now we will prove the first existence for Eq.(1) using Kranselskii fixed point theorem .

Theorem 3.1. Assume (H₁–H₄) satisfied. Then Eq.1 has atleast one solution in $C_{1-\gamma}^{\gamma}[a, b] \subset C_{1-\gamma}^{\alpha, \beta}[a, b]$.

Proof. According to Lemma 2.12, it is sufficient to prove the existence result for the mixed type integral Eq.(3) Consider the operator $\Omega : C_{1-\gamma}[a, b] \rightarrow C_{1-\gamma}[a, b]$ given by

$$\begin{aligned} (\Omega)x(t) &= \frac{T}{\Gamma(\alpha)}(t-a)^{\gamma-1} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds. \end{aligned} \quad (10)$$

It is obvious that the operator Ω is well defined.

Set $\tilde{f}(s) = f(s, 0)$ and

$$\omega = \frac{B(\gamma, \alpha)}{\Gamma(\alpha)} \left(|T| \sum_{i=1}^m \lambda_i (\tau_i - a)^{\alpha+\gamma-1} + (b-a)^\alpha \right) \|\tilde{f}\|_{C_{1-\gamma}}.$$

Consider a ball

$$B_r = \{x \in C_{1-\gamma}[a, b] : \|x\|_{C_{1-\gamma}} \leq r\} \quad \text{with} \quad r \geq \frac{\omega}{1-\varrho} (\varrho < 1).$$

Next, we subdivide the operator Ω into two operators P and Q on B_r , as follows.

$$\begin{aligned} (Px)(t) &= \frac{T}{\Gamma(\alpha)}(t-a)^{\gamma-1} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds \\ (Qx)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds. \end{aligned}$$

The proof is divided into several steps.

Step 1. $Px + Qy \in B_r$ for every $x, y \in B_r$.

The operator P ,

$$(Px)(t)(t-a)^{1-\gamma} = \frac{T}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds$$

then,

$$\begin{aligned} |(Px)(t)(t-a)^{1-\gamma}| &\leq \frac{|T|}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} |f(s, x(s), T(x(s)), S(x(s)))| ds \\ &\leq \frac{|T|}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} \\ &\quad \times |f(s, x(s), T(x(s)), S(x(s))) - f(s, 0)| + f(s, 0) ds \\ &\leq \frac{|T|}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} \\ &\quad \times \left(L_1|x(s)| + L_2|x(s)| + L_3|x(s)| + |\tilde{f}(s)| \right) ds \\ &\leq \frac{|T|}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} \left(L|x(s)| + |\tilde{f}(s)| \right) ds \\ &\leq \frac{|T|}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \left[(\tau_i - s)^{\alpha+\gamma-1} B(\gamma, \alpha) \left(L\|x\|_{C_{1-\gamma}} + \|\tilde{f}\|_{C_{1-\gamma}} \right) \right]. \end{aligned}$$

Where,

$$\begin{aligned} \int_a^t (t-s)^{\alpha-1} |x(s)| ds &\leq \left(\int_a^t (t-s)^{\alpha-1} (s-a)^{\gamma-1} ds \right) \|x\|_{C_{1-\gamma}} \\ &= (t-a)^{\alpha+\gamma-1} B(\gamma, \alpha) \|x\|_{C_{1-\gamma}} \end{aligned} \quad (11)$$

this gives

$$\|Px\|_{C_{1-\gamma}} \leq \frac{|T|B(\gamma, \alpha)}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \left[(\tau_i - s)^{\alpha+\gamma-1} \left(L\|x\|_{C_{1-\gamma}} + \|f\|_{C_{1-\gamma}} \right) \right]. \quad (12)$$

For the operator Q,

$$(Qx)(t)(t-a)^{1-\gamma} \leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (t-a)^{1-\gamma} f(s, x(s), T(x(s)), S(x(s))) ds$$

Then we proceed as (12),

$$\begin{aligned} |(Qx)(t)(t-a)^{1-\gamma}| &\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (t-a)^{1-\gamma} |f(s, x(s), T(x(s)), S(x(s)))| ds \\ &\leq \frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left(L_1|x(s)| + L_2|x(s)| + L_3|x(s)| + |\tilde{f}(s)| \right) ds \\ &\leq \frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left(L|x(s)| + |\tilde{f}(s)| \right) ds \\ &\leq \frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)} \left[\lambda_i (t-a)^{\alpha+\gamma-1} B(\gamma, \alpha) \left(L\|x\|_{C_{1-\gamma}} + \|\tilde{f}\|_{C_{1-\gamma}} \right) \right] \\ &\leq \frac{B(\gamma, \alpha)}{\Gamma(\alpha)} (b-a)^\alpha \left(L\|x\|_{C_{1-\gamma}} + \|\tilde{f}\|_{C_{1-\gamma}} \right). \end{aligned}$$

Thus,

$$\|Qx\|_{C_{1-\gamma}} \leq \frac{B(\gamma, \alpha)}{\Gamma(\alpha)} (b-a)^\alpha \left(L\|x\|_{C_{1-\gamma}} + \|\tilde{f}\|_{C_{1-\gamma}} \right). \quad (13)$$

Combining Eq.(12) and (13), for every $x, y \in B_r$,

$$\|Px + Qy\|_{C_{1-\gamma}} \leq \|Px\|_{C_{1-\gamma}} + \|Qy\|_{C_{1-\gamma}} \leq \varrho r + \omega \leq r,$$

which implies that $Px + Qy \in B_r$.

Step 2. The operator P is contraction mapping.

For the operator P, any $x, y \in B_r$

$$\begin{aligned} |((Px)(t) + (Py)(t))(t-a)^{1-\gamma}| &= \frac{|T|}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{\alpha-1} |f(s, x(s), T(x(s)), S(x(s))) \\ &\quad - f(s, y(s), T(y(s)), S(y(s)))| ds \\ &\leq \frac{|T|}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_a^{\tau_i} (\tau_i - s)^{1-\alpha} \\ &\quad \times \left(L_1\|x - y\| + L_2\|x - y\| + L_3\|x - y\| \right) ds. \end{aligned} \quad (14)$$

Now,

$$\begin{aligned}
 \int_a^t L_2 \|Tx - Ty\| ds &= L_2 \int_a^t \int_a^s \|K(s, t)\| \|x(\tau) - y(\tau)\| d\tau ds \\
 &\leq L_2 \int_a^t \|x(s) - y(s)\| \int_a^s \|K(s, t)\| d\tau ds \\
 &\leq L_2 \|x(s) - y(s)\| \int_a^t K^* ds \\
 &\leq L_2 \|x - y\|_{C_{1-\gamma}} K^*.
 \end{aligned} \tag{15}$$

$$\begin{aligned}
 |(Px)(t) - (Py)(t)|(t-a)^{1-\gamma} &\leq \frac{|T|B(\gamma, \alpha) \sum_{i=1}^m \lambda_i (\tau_i - s)^{\alpha+\gamma-1}}{\Gamma(\alpha)} \\
 &\quad \times \left(L_1 \|x - y\|_{C_{1-\gamma}} + L_2 \|x - y\|_{C_{1-\gamma}} + L_3 \|x - y\|_{C_{1-\gamma}} \right) \\
 &\leq \frac{|T|B(\gamma, \alpha) \sum_{i=1}^m \lambda_i (\tau_i - s)^{\alpha+\gamma-1}}{\Gamma(\alpha)} \\
 &\quad \times \left(L_1 + L_2 K^* + L_3 H^* \right) \|x - y\|_{C_{1-\gamma}}.
 \end{aligned}$$

Using definition of L, we can write,

$$\begin{aligned}
 &\leq \frac{|T|B(\gamma, \alpha) \sum_{i=1}^m \lambda_i (\tau_i - s)^{\alpha+\gamma-1}}{\Gamma(\alpha)} \\
 &\quad \times \left(1 + K^* + H^* \right) \|x - y\|_{C_{1-\gamma}}.
 \end{aligned}$$

By (H_3) and (H_4)

$$\begin{aligned}
 \|Px - Py\|_{C_{1-\gamma}} &\leq \frac{|T|B(\gamma, \alpha) \sum_{i=1}^m \lambda_i (\tau_i - s)^{\alpha+\gamma-1}}{\Gamma(\alpha)} \|x - y\|_{C_{1-\gamma}} \\
 &\leq \varrho \|x - y\|_{C_{1-\gamma}}.
 \end{aligned}$$

Therefore the operator P is contraction by (H_2) .

Step 3. The operator Q is compact and continuous.

Since the function $f \in C_{1-\gamma}[[a, b]]$, the operator Q is continuous following the definition of $C_{1-\gamma}[[a, b]]$. According to step.1,

$$\|Qx\|_{C_{1-\gamma}} \leq \frac{B(\gamma, \alpha)}{\Gamma(\alpha)} (b-a)^\alpha \left(L \|x\|_{C_{1-\gamma}} + \|\bar{f}\|_{C_{1-\gamma}} \right), \tag{16}$$

so Q is uniformly bounded on B_r .

To prove the compactness, for any $a < t_1 < t_2 \leq b$,

$$\begin{aligned}
 |Qx(t_1) - Qx(t_2)| &= \left| \frac{1}{\Gamma(\alpha)} \int_a^{t_1} (t_1 - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds \right. \\
 &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_a^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s), T(x(s)), S(x(s))) ds \right| \\
 &\leq \frac{\|f\|_{C_{1-\gamma}}}{\Gamma(\alpha)} \left| \int_a^{t_1} (t_1 - s)^{\alpha-1} (s - a)^{\gamma-1} ds - \int_a^{t_2} (t_2 - s)^{\alpha-1} (s - a)^{\gamma-1} ds \right| \\
 &\leq \frac{\|f\|_{C_{1-\gamma}} B(\gamma, \alpha)}{\Gamma(\alpha)} \left| (t_1 - a)^{\alpha+\gamma-1} - (t_2 - a)^{\alpha+\gamma-1} \right|.
 \end{aligned}$$

tending to zero as $t_2 \rightarrow t_1$ whether $\alpha + \gamma < 1$ or $\alpha + \gamma \geq 1$. Hence Q is equicontinuous on B_r by Arzela-Ascoli theorem.

If follows Krasnoselskii fixed point theorem the problem (1) has atleast one solution $x \in C_{1-\gamma}[a, b]$. Finally using Lemma 2.10 and repeating the process of proof in Lemma 2.12 or [7], we can show that the solution in $C_{1-\gamma}^\gamma[a, b]$. This completes the proof. $\square$

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