

Square Difference labeling for Complete Bipartite Graphs and Trees

S. Vidyanandini¹ and N. Parvathi ¹

¹Department of Mathematics,
SRM University, Chennai, Tamil Nadu
vidhyanandhini.maths@gmail.com

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Abstract

Let G be a graph with p vertices and q edges. A square difference labeling of a graph G is a bijection $f : V(G) \rightarrow \{0, 1, 2, \dots, p-1\}$, such that the induced function $f^* : E(G) \rightarrow N$ defined by $f^*(uv) = |f(u)^2 - f(v)^2|$ for every edge $uv \in E(G)$ are all distinct. A graph which admits square difference labeling is called a square difference labeling graph. In this paper, we show that complete bipartite graphs $K_{m,n}$, admit square difference labeling.

AMS Subject Classification: ...

Key Words and Phrases: Square difference Labeling, Bipartite graphs, Tree.

1 Introduction

All graph considered here are finite and simple. For the definitions not defined in this paper refer [5]. Shiama[4,6] proved that cycle, complete graph, cycle cactus, ladder, lattice grids, wheels, quadrilateral snakes, the graph $G = K_2 + mK_1$, central graphs of path, square graphs of path, some path related graphs, fan and gear graphs are all square difference labeling graphs [6].

The square sum labeling and square sum graphs [1] are defined and discussed by V.Ajitha, S.Arumugam and K.A.Germina. They

proved that the cycle, complete graph K_n , the cycle cactus, ladder and complete lattice grids are square sum graphs. In a similar way L.Beineke and S.M.Hegde[2] proved that some graphs are strong multiplicatively indexable. The square sum labeling for some central and total graphs [6,7] is taken for references. For the complete survey on square difference labeling, refer excellent dynamic survey by Gallian [3]. In this paper we prove that complete bipartite graphs $K_{m,n}$ admit square difference labeling.

Definition 1 Let $G = (V(G), E(G))$ be a graph G is said to be square difference labeling if there exists a bijection $f : V(G) \rightarrow \{0, 1, 2, \dots, p-1\}$ such that the induced function $f^* : E(G) \rightarrow N$ given by $f^*(uv) = |[f(u)]^2 - [f(v)]^2|$ for every $uv \in E(G)$ are all distinct. Any graph which admits square difference labeling is called a square difference labeling graph[7].

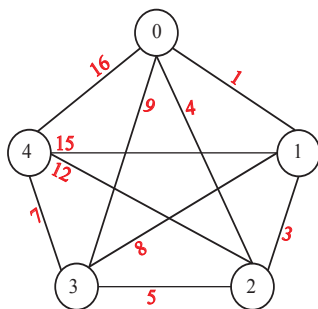


Figure 1: Square difference labeling for K_5

2 Main Result:

In this section, we prove that complete bipartite graphs $K_{m,n}$ admits square difference labeling.

Theorem 1. *The complete bipartite graph $K_{m,n}$ for any integer $m, n > 0$ admits square difference labeling.*

Proof. Let G be a complete bipartite graph $K_{m,n}$ for any positive integer m and n . Note that by the definition of complete

bipartite graph, $K_{m,n}$ has $m + n$ vertices and mn edges, Without loss of generality, let us assume that $m \leq n$.

Let $|V_1| = m$ and $|V_2| = n$.

Let the vertex subset V_1 has $\{v_0, v_1, \dots, v_{m-1}\}$ and the vertex subset V_2 has $\{u_0, u_1, \dots, u_{n-1}\}$. Define the vertex labeling $f : V_1 \cup V_2 \rightarrow \{0, 1, 2, \dots, (m + n) - 1\}$ as $f(v_i) = i$ for $0 \leq i \leq m - 1$ and $f(u_j) = m + j$ for $0 \leq j \leq n - 1$. Define the edge labeling f^* as $f^*(uv) = | [f(u)]^2 - [f(v)]^2 |$, for any edge $uv \in E(G)$. From the definition of function f , it is clear that vertex labels of G are distinct and function f is bijective.

□

Claim: The edge labels of the edges of G are distinct.

Let u_j and u_{j+1} for $j, 0 \leq j \leq n - 2$, be the vertices in V_2 , such that their vertex labels are consecutive. Let us assume that $f(u_j) = r$ and $f(u_{j+1}) = r + 1$. By the definition of vertex labels of vertices in V_2 , it is clear that $r \geq m$.

Since G is a complete bipartite graph, vertex u_j is adjacent to every vertex v_i in V_1 , for $i, 0 \leq i \leq m - 1$. Similarly, vertex u_{j+1} is adjacent to every vertex v_i in V_1 , for $i, 0 \leq i \leq m - 1$. Since the vertex label of v_i and the vertex label of u_j are of distinct, the induced edge labels of the edges $u_j v_i$, for $i, 0 \leq i \leq m - 1$ are distinct. Similarly, the induced edge labels of the edges $u_{j+1} v_i$, for $i, 0 \leq i \leq m - 1$ are distinct. Further, the induced edge label of the edges $u_j v_i$ form a monotonically increasing sequence as i increases from 0 to $m - 1$, Similarly, the induced edge label of the edges $u_{j+1} v_i$ form a monotonically increasing sequence as i increases from 0 to $m - 1$. By the definition of f and f^* . Now, we show that the induced edge label for the edge $u_{j+1} v_{m-1}$ is strictly greater than the induced edge label for the edge $u_j v_0$.

$$\begin{aligned} f^*(u_j v_0) &= | [f(u_j)]^2 - [f(v_0)]^2 | \\ &= | r^2 - 0^2 | \\ &= r^2 \end{aligned}$$

$$\begin{aligned}
 f^*(u_{j+1}v_{m-1}) &= | [f(u_{j+1})]^2 - [f(v_{m-1})]^2 | \\
 &= | (r+1)^2 - (m-1)^2 | \\
 &= r^2 + 2r + 1 - m^2 - 1 + 2m \\
 &= r^2 + 2r + 2m - m^2 \\
 &> 0 (\text{Since } r \geq m)
 \end{aligned}$$

Thus, the edge labels of the G are distinct. Therefore, the com-

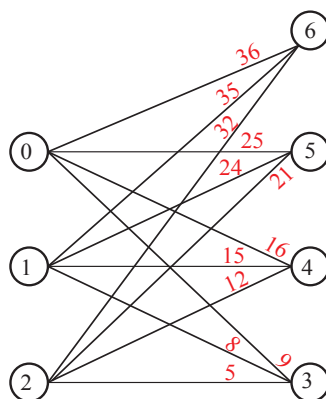


Figure 2: Square difference labeling for $K_{3,4}$

plete bipartite graphs $G = K_{m,n}$ for any integer $m, n > 0$ admits square difference labeling.

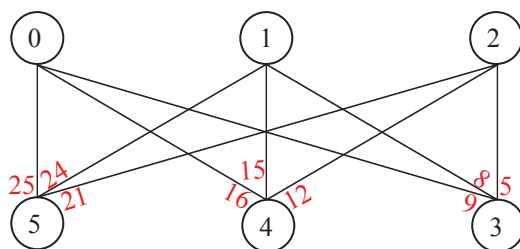


Figure 3: Square difference labeling for $K_{3,3}$

Theorem 2. Any tree T with m edges admit square difference labeling.

Proof. Let T be a tree with m edges and $m + 1$ vertices. Since all trees are bipartite graphs, consider the bipartition of the vertex set $V(T)$.

$$\text{Let } V(T) = V_1 \cup V_2$$

Let $|V_1| = p$. Therefore $|V_2| = (m + 1) - p$. Let the vertices in V_1 be $u_0, u_1, \dots, u_{p-1}$ and order the vertices in V_2 as $u_p, u_{p+1}, \dots, u_m$ in the bottom to top order as defined by the bipartition of vertices of tree.

Define the vertex labeling function $f(u_i) = i$, for $i, 0 \leq i \leq m$. and define the edge labeling function.

$f^*(uv) = |f(u)|^2 - |f(v)|^2$ | from the definition of f , it is clear that vertex labels of G are distinct is bijective. $\square$

Claim: The edge labels of the edges of T are distinct.

Let u_j and u_{j+1} for $i, 0 \leq j \leq n - 2$, be the vertices in V_2 , such that their vertex labels are consecutive. Let us assume that $f(u_i) = r$ and $f(u_{j+1}) = r + 1$. By the definition of vertex labels of vertices in V_2 , it is clear that $r \geq m$. Since T is a tree, consider the adjacent vertices of vertex u_j . Among the adjacent vertices of u_j , consider the vertex (say u_α) with maximum label. By the definition of vertex labeling, $f(u_\alpha) = \alpha$. Now, consider the adjacent vertices of vertex u_{j+1} . Among the adjacent vertices of u_{j+1} , consider the vertex (say u_β) with maximum label. By the definition of vertex labeling, $f(u_\beta) = \beta$. Thus, it is enough to prove that the induced edge of the edge $u_j u_\alpha$ is strictly less than the induced edge label of the edge $u_{j+1} u_\beta$.

Case:1 $f(u_\alpha) = \alpha > f(u_\beta) = \beta$, By the definition of f and f^* .

$$\begin{aligned} & f^*(u_{j+1}u_\beta) - f^*(u_ju_\alpha) \\ &= |[f(u_{j+1})]^2 - [f(u_\beta)]^2| - |[f(u_j)]^2 - [f(u_\alpha)]^2| \\ &= |(r+1)^2 - \beta^2| - |r^2 - \alpha^2| \\ &= (r+1)^2 - r^2 + \alpha^2 - \beta^2 \\ &> 0 (\text{since } \alpha > \beta) \end{aligned}$$

Case:2 $f(u_\beta) = \beta > f(u_\alpha) = \alpha$, By the definition of f and f^* , Let

$$\beta = \alpha + s, s \geq 1.$$

$$\begin{aligned} & f^*(u_{j+1}u_\beta) - f^*(u_ju_\alpha) \\ &= |[f(u_{j+1})]^2 - [f(u_\beta)]^2| - |[f(u_j)]^2 - [f(u_\alpha)]^2| \\ &= (r+1)^2 - \beta^2 - [r^2 - \alpha^2] \\ &= (r+1)^2 - r^2 - \beta^2 + \alpha^2 \\ &= 2r + 1 + \alpha^2 - [\alpha^2 + s^2 + 2\alpha s] \\ &= 2r - 2\alpha s - s^2 \\ &> 0 (\text{since } r > \alpha \text{ \& } r > s) \end{aligned}$$

Case:3 $f(u_\alpha) = \alpha = f(u_\beta) = \beta$, By the definition of f and f^* .

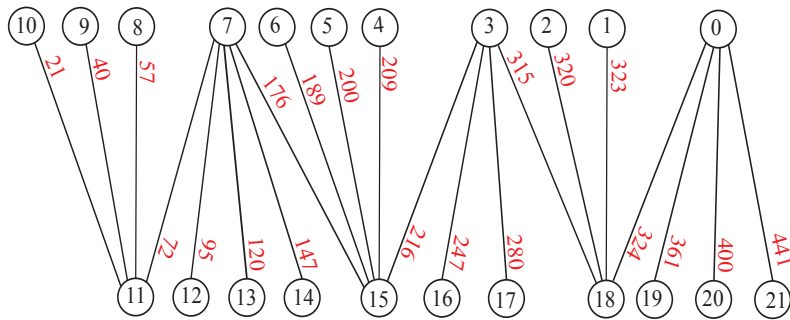


Figure 4: Square difference labeling for Tree T with 21 edges.

$$\begin{aligned} & f^*(u_{j+1}u_\beta) - f^*(u_ju_\alpha) \\ &= |[f(u_{j+1})]^2 - [f(u_\beta)]^2| - |[f(u_j)]^2 - [f(u_\alpha)]^2| \\ &= (r+1)^2 - \beta^2 - [r^2 - \alpha^2] \\ &= r^2 + 1 + 2r - r^2 \\ &= 2r + 1 \\ &> 0 \end{aligned}$$

Thus, the edge labels of the edges of T are distinct. Therefore, any tree T with m edges admits square difference labeling.

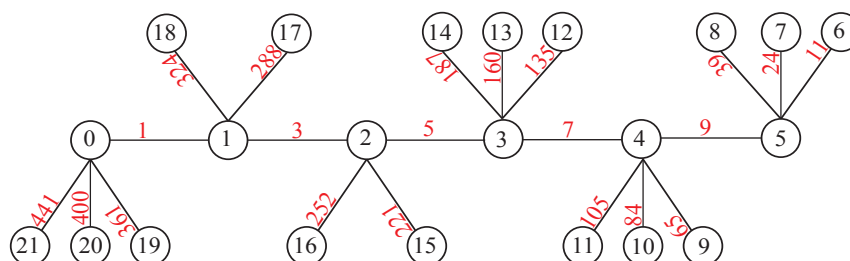


Figure 5: Square difference labeling for Tree T with 21 edges.

3 Conclusion and Discussion

In this paper, we proved the generalized results that $K_{m,n}$, for $m, n > 0$ and any tree T with m edges admit square difference labeling. Based on the truthness of the above two results, we pose a related question that “Whether complete tripartite graphs $K_{m,n,r}$ for $m, n, r > 0$ admits Square difference labeling ? ”.

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