

Kamenev-type oscillation criteria for generalized sublinear delay difference equations

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ABSTRACT. Using Riccati transformation techniques, we present some new oscillation criteria for generalized second order nonlinear difference equation

$$\Delta_{\ell}(p(k\ell + j)(\Delta_{\ell}(u(k\ell + j)))^{\gamma}) + q(k\ell + j)u^{\beta}((k - \sigma)\ell + j) = 0,$$

when $0 < \beta < 1$ is a quotient of odd positive integers, $k \in (0, \infty)$, $\ell \in (0, \infty)$, $j = k - \lfloor \frac{k}{\ell} \rfloor \ell$.

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1. INTRODUCTION

Difference equations is a captivating mathematical area on its own as well as a rich field of the applications in such various disciplines such as population dynamics, operations research, ecology, economics, biology etc. For general background as difference equations with numerous examples from various fields, one can refer to [1].

The study of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k)$, $k \in \mathbb{N} = \{0, 1, 2, \dots\}$. Eventhough many authors [1], [8] have suggested the definition of Δ as

$$\Delta u(k) = u(k + \ell) - u(k), \quad \ell \in (0, \infty), \quad (1)$$

no major progress have taken place on this line.

In [3], authors took up the definition of Δ as given in (1), they produced some important results in difference equations. For convenience, we labelled the operator Δ defined by (1) as Δ_{ℓ} and by defining its inverse Δ_{ℓ}^{-1} , many interesting results in number theory were obtained. By extending theory of Δ_{ℓ} to complex function, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were established for the solutions of difference equations involving Δ_{ℓ} . The results obtained can be found in [3, 4, 6, 7].

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Recent, the asymptotic behavior of second order difference equations has been the investigate by many authors [9, 10].

With this back ground, we will be studying the generalized second order sublinear delay difference equations of the form

$$\Delta_{\ell}(p(k\ell + j)(\Delta_{\ell}(u(k\ell + j)))^{\gamma}) + q(k\ell + j)u^{\beta}((k - \sigma)\ell + j) = 0, \quad (2)$$

where Δ_{ℓ} denotes the forward difference operator for any real valued function $u(k\ell + j)$, $k \in (o, \infty)$, $\ell \in (0, \infty)$, $j = k - [\frac{k}{\ell}] \ell$, γ is quotient of odd positive integers, $0 < \beta < 1$ is quotient of odd positive integers, σ is a fixed nonnegative integer, $p(k\ell + j) > 0$, and $q(k\ell + j) \geq 0$ are real valued functions, and for some $k_0 > 0$,

$$\sum_{k=k_0}^{\infty} \left(\frac{1}{p(k\ell + j)} \right)^{1/\gamma} = \infty, \quad (3)$$

By a solution of (2) we mean a nontrivial real valued function $u(k)$ defined for $k \geq -\sigma$, and satisfies equation (2) for $k \in (0, \infty)$. Clearly, if

$$u(k\ell + j) = A(k\ell + j) \text{ for } k \in [-\sigma, 0] \quad (4)$$

are given, then equation (2) has a unique solution satisfying the initial condition (4). A solution $u(k\ell + j)$ of (2) is said to be oscillatory if for every $k_1 > 0$ there exists an $k \geq k_1$ such that $u(k\ell + j)u((k + 1)\ell + j) \leq 0$, otherwise it is nonoscillatory. Equation (2) is said to be oscillatory if all its solutions are oscillatory.

2. MAIN RESULTS

Theorem 2.1. Assume that (3) holds. Furthermore, assume that there exists a positive real valued functions $\rho(k\ell + j)$ such that for every $\eta \geq 1$ and positive number M .

$$\limsup_{k \rightarrow \infty} \sum_{l=k_0}^k [\rho(l\ell + j)q(l\ell + j) - \frac{(p((l - \sigma)\ell + j))^{1/\gamma} \eta^{1-\beta} ((l - \sigma + 1)\ell + j)^{1-\beta} (\Delta_{\ell} \rho(l\ell + j))^2}{4\beta(M)^{(\gamma-1)/\gamma} \rho(l\ell + j)}] \quad (5)$$

Then every solution of equation (2) oscillates.

Proof. Suppose to the contrary that $u(k\ell + j)$ is an eventually nonoscillatory solution of (2) such that $u((k-\sigma)\ell + j) > 0$ for all $k \geq k_0 > 0$. We shall consider only this case, since the substitution $v(k\ell + j) = -u(k\ell + j)$ transforms equation (2) into an equation of the same form. From equation (2) we have

$$\Delta_\ell(p(k\ell + j)(\Delta_\ell u(k\ell + j))^\gamma) = -q(k\ell + j)u^\beta((k-\sigma)\ell + j) \leq 0, \quad k \geq k_0 \quad (6)$$

and so $p(k\ell + j)(\Delta_\ell u(k\ell + j))^\gamma$ is an eventually nonincreasing sequence. We first show that $p(k\ell + j)(\Delta_\ell u(k\ell + j))^\gamma \geq 0$ for $k \geq k_0$. In fact, if there exists a real $k_1 \geq k_0$ such that $p(k_1\ell + j)(\Delta_\ell u(k_1\ell + j))^\gamma = c < 0$, then (6) implies that $p(k\ell + j)(\Delta_\ell u(k\ell + j))^\gamma \leq c$ for $k \geq k_1$ that is

$$\Delta_\ell u(k\ell + j) \leq \left(\frac{c}{p(k\ell + j)} \right)^{1/\gamma}$$

and, hence

$$u(k\ell + j) \leq u(k_1\ell + j) + c^{1/\gamma} \sum_{i=k_1}^{k-1} \left(\frac{1}{p(i\ell + j)} \right)^{1/\gamma} \rightarrow -\infty \text{ as } k \rightarrow \infty \quad (7)$$

which contradicts the fact that $u(k\ell + j) > 0$ for $k \geq k_0$, then $p(k\ell + j)(\Delta_\ell u(k\ell + j))^\gamma \geq 0$. Also we claim that $\Delta_\ell^2 u(k\ell + j) \leq 0$. If not there exists $k_1 \geq k_0$ such that $\Delta_\ell^2 u(k\ell + j) > 0$ for $k \geq k_1$ and this implies that $\Delta_\ell u((k+1)\ell + j) > \Delta_\ell u(k\ell + j)$, so that since $\Delta_\ell p(k\ell + j) \geq 0$, $p((k+1)\ell + j)(\Delta_\ell u((k+1)\ell + j))^\gamma > p((k+1)\ell + j)(\Delta_\ell u(k\ell + j))^\gamma \geq p(k\ell + j)(\Delta_\ell u(k\ell + j))^\gamma$ and this contradicts the fact that $p(k\ell + j)(\Delta_\ell u(k\ell + j))^\gamma$ is nonincreasing sequence, then $\Delta_\ell^2 u(k\ell + j) \leq 0$, and therefore we have

$$u(k\ell + j) > 0, \Delta_\ell u(k\ell + j) \text{ and } \Delta_\ell^2 u(k\ell + j) \leq 0 \text{ for } k \geq k_0. \quad (8)$$

Define the sequence $z(k\ell + j)$ by

$$z(k\ell + j) = \rho(k\ell + j) \frac{p(k\ell + j)(\Delta_\ell u(k\ell + j))^\gamma}{u^\beta((k-\sigma)\ell + j)} \quad (9)$$

then $z(k\ell + j) > 0$, and

$$\begin{aligned} \Delta_\ell z(k\ell + j) &= p((k+1)\ell + j)(\Delta_\ell u((k+1)\ell + j)) \Delta_\ell \left(\frac{\rho(k\ell + j)}{u^\beta((k-\sigma)\ell + j)} \right) \\ &+ \frac{\rho(k\ell + j) \Delta_\ell(p(k\ell + j)(\Delta_\ell u(k\ell + j))^\gamma)}{u^\beta((k-\sigma)\ell + j)} \end{aligned} \quad (10)$$

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From (2) and (10), we have

$$\begin{aligned} \Delta_\ell z(k\ell + j) &= -\rho(k\ell + j)q(k\ell + j) + \frac{\Delta_\ell \rho(k\ell + j)}{\rho((k+1)\ell + j)} z((k+1)\ell + j) \\ &\quad - \frac{\rho(k\ell + j)p((k+1)\ell + j)(\Delta_\ell u((k+1)\ell + j))^\gamma \Delta_\ell u^\beta((k-\sigma)\ell + j)}{u^\beta((k-\sigma+1)\ell + j)u^\beta((k-\sigma)\ell + j)} \end{aligned} \quad (11)$$

From (6) and (8), we get

$$\begin{aligned} p((k-\sigma)\ell + j)(\Delta_\ell u((k-\sigma)\ell + j))^\gamma &\geq p((k+1)\ell + j)(\Delta_\ell u((k+1)\ell + j))^\gamma \\ \text{and } u((k+1-\sigma)\ell + j) &\geq u((k-\sigma)\ell + j) \end{aligned} \quad (12)$$

and then from (11) and (12), we have

$$\begin{aligned} \Delta_\ell z(k\ell + j) &\leq -\rho(k\ell + j)q(k\ell + j) + \frac{\Delta_\ell \rho(k\ell + j)}{\rho((k+1)\ell + j)} z((k+1)\ell + j) \\ &\quad - \frac{\rho(k\ell + j)p((k+1)\ell + j)(\Delta_\ell u((k+1)\ell + j))^\gamma \Delta_\ell u^\beta((k-\sigma)\ell + j)}{(u^\beta((k-\sigma+1)\ell + j))^2} \end{aligned} \quad (13)$$

Now, by using the inequality in [2]

$$u^\beta - v^\beta \geq \beta u^{\beta-1}(u-v) \text{ for all } u \neq v > 0 \text{ and } 0 < \beta \leq 1.$$

Then, we have

$$\Delta_\ell u^\beta((k-\sigma)\ell + j) = \beta(u((k-\sigma+1)\ell + j))^{\beta-1} \Delta_\ell u((k-\sigma)\ell + j). \quad (14)$$

Substitute from (14) in (13), we have

$$\begin{aligned} \Delta_\ell z(k\ell + j) &\leq -\rho(k\ell + j)q(k\ell + j) + \frac{\Delta_\ell \rho(k\ell + j)}{\rho((k+1)\ell + j)} z((k+1)\ell + j) \\ &\quad - \left(\frac{\rho(k\ell + j)p((k+1)\ell + j)\beta(u((k+1-\sigma)\ell + j))^{\beta-1}}{(u^\beta((k-\sigma+1)\ell + j))^2} \right) \\ &\quad \times \Delta_\ell u((k-\sigma)\ell + j)(\Delta_\ell u((k+1)\ell + j))^\gamma \end{aligned} \quad (15)$$

From (12) and (15), we have

$$\begin{aligned} \Delta_\ell z(k\ell + j) &\leq -\rho(k\ell + j)q(k\ell + j) + \frac{\Delta_\ell \rho(k\ell + j)}{\rho((k+1)\ell + j)} z((k+1)\ell + j) \\ &\quad - \frac{\beta \rho(k\ell + j)(p((k+1)\ell + j))^{1/\gamma} p((k+1)\ell + j)(\Delta_\ell u((k+1)\ell + j))^{\gamma+1}}{(p((k-\sigma)\ell + j))^{1/\gamma} (u((k-\sigma+1)\ell + j))^{1-\beta} (u^\beta((k-\sigma+1)\ell + j))} \end{aligned}$$

hence,

$$\begin{aligned} \Delta_{\ell} z(k\ell + j) &\leq -\rho(k\ell + j)q(k\ell + j) + \frac{\Delta_{\ell}\rho(k\ell + j)}{\rho((k+1)\ell + j)} z((k+1)\ell + j) \\ &\quad - \left(\frac{\beta\rho(k\ell + j)(p((k+1)\ell + j))^{(1/\gamma)-1}}{(p((k+1)\ell + j))^2(p((k-\sigma)\ell + j))^{1/\gamma}(u((k-\sigma+1)\ell + j))^{1-\beta}} \right. \\ &\quad \times \left. \frac{(p((k+1)\ell + j))^2(\rho((k+1)\ell + j))^2(\Delta_{\ell}u((k+1)\ell + j))^{2\gamma}}{(u^{\beta}((k-\sigma+1)\ell + j))^2(\Delta_{\ell}u((k+1)\ell + j))^{\gamma-1}} \right) \end{aligned} \quad (16)$$

From (8), we conclude that

$$u(k\ell + j) \leq u(k_0\ell + j) + \Delta_{\ell}u(k_0\ell + j)((k - k_0)\ell + j), k \geq k_0$$

thus there exists a $k_1 \geq k_0$ and by choice of constant $\eta \geq 1$ such that

$$u(k\ell + j) \leq \eta(k\ell + j) \text{ for } k \geq k_1$$

and this implies that

$$u((k - \sigma + 1)\ell + j) \leq \eta((k - \sigma + 1)\ell + j) \text{ for } k \geq k_2 = k_1 + \sigma - 1$$

and, hence

$$\frac{1}{(u((k - \sigma + 1)\ell + j))^{1-\beta}} \geq \frac{1}{(\eta((k - \sigma + 1)\ell + j))^{1-\beta}} \quad (17)$$

Since $p(k\ell + j)(\Delta_{\ell}u(k\ell + j))^{\gamma}$ is a positive and increasing function, there exists a $k_2 \geq k_1$ sufficiently large such that $p(k\ell + j)(\Delta_{\ell}u(k\ell + j))^{\gamma} \leq 1/M$ for some positive constant M and $k \geq k_2$, and hence by (6) we have $p((k+1)\ell + j)(\Delta_{\ell}u((k+1)\ell + j))^{\gamma} \leq 1/M$, so that

$$\frac{1}{(\Delta_{\ell}u((k+1)\ell + j))^{\gamma-1}} \geq (Mp((k+1)\ell + j))^{(\gamma-1)/\gamma} \quad (18)$$

then from (9), (16), (17) and (18) we have

$$\begin{aligned} \Delta_{\ell} z(k\ell + j) &\leq -\rho(k\ell + j)q(k\ell + j) + \frac{\Delta_{\ell}\rho(k\ell + j)}{\rho((k+1)\ell + j)} z((k+1)\ell + j) \\ &\quad - \frac{\beta\rho(k\ell + j)M^{(\gamma-1)/\gamma}(z((k+1)\ell + j))^2}{(p((k+1)\ell + j))^2(p((k-\sigma)\ell + j))^{1/\gamma}\eta^{1-\beta}((k-\sigma+1)\ell + j)^{1-\beta}} \end{aligned} \quad (19)$$

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Hence,

$$\begin{aligned}
\Delta_\ell z(k\ell + j) &\leq -\rho(k\ell + j)q(k\ell + j) + \\
&+ \frac{(p((k - \sigma)\ell + j))^{1/\gamma} \eta^{1-\beta} ((k - \sigma + 1)\ell + j)^{1-\beta} (\Delta_\ell \rho(k\ell + j))^2}{4\beta(M)^{(\gamma-1)/\gamma} \rho(k\ell + j)} \\
&- \left[\frac{\sqrt{\beta(M)^{(\gamma-1)/\gamma} \rho(k\ell + j)} z((k + 1)\ell + j)}{\rho((k + 1)\ell + j) \sqrt{\eta((k - \sigma + 1)\ell + j)^{1-\beta} p((k - \sigma)\ell + j)}} \right. \\
&\quad \left. \frac{\sqrt{\eta^{1-\beta} ((k - \sigma + 1)\ell + j)^{1-\beta} (p((k - \sigma)\ell + j))^{1/\gamma} \Delta_\ell \rho(k\ell + j)}}{2\sqrt{\beta(M)^{(\gamma-1)/\gamma} \rho(k\ell + j)}} \right]^2 \\
&< -[\rho(k\ell + j)q(k\ell + j) \\
&\quad - \frac{(p((k - \sigma)\ell + j))^{1/\gamma} \eta^{1-\beta} ((k - \sigma + 1)\ell + j)^{1-\beta} (\Delta_\ell \rho(k\ell + j))^2}{4\beta(M)^{(\gamma-1)/\gamma} \rho(k\ell + j)}] \quad (20)
\end{aligned}$$

Summing (20) from k_2 to k , we obtain

$$\begin{aligned}
&-z(k_2\ell + j) < z((k + 1)\ell + j) - z(k_2\ell + j) \\
&< -\sum_{l=k_2}^k [\rho(l\ell + j)q(l\ell + j) \\
&\quad - \frac{(p((l - \sigma)\ell + j))^{1/\gamma} \eta^{1-\beta} ((l - \sigma + 1)\ell + j)^{1-\beta} (\Delta_\ell \rho(l\ell + j))^2}{4\beta(M)^{(\gamma-1)/\gamma} \rho(l\ell + j)}]
\end{aligned}$$

which yields

$$\sum_{l=k_2}^k \left[\rho(l\ell + j)q(l\ell + j) - \frac{(p((l - \sigma)\ell + j))^{1/\gamma} \eta^{1-\beta} ((l - \sigma + 1)\ell + j)^{1-\beta} (\Delta_\ell \rho(l\ell + j))^2}{4\beta(M)^{(\gamma-1)/\gamma} \rho(l\ell + j)} \right] < c_1$$

for all large k , and this is contrary to (5). The proof is complete. $\square$

Remark 2.2. Note that from Theorem 2.1, we can obtain different conditions for oscillations for oscillation of all solutions of equation (2) when (3) holds by different choices of $\rho(k\ell + k)$.

Remark 2.3. When $\gamma = \beta = 1$, equation (2) reduces to the generalized linear delay difference equation

$$\Delta_\ell(p(k\ell + j)\Delta_\ell u((k - \sigma)\ell + j)) + q(k\ell + j)u((k - \sigma)\ell + j) = 0, \quad (21)$$

where $k \in (0, \infty)$, $\ell \in (0, \infty)$ and the conditions (5) in Theorem 2.1 reduces to

$$\limsup_{k \rightarrow \infty} \sum_{l=k_0}^k \left[\rho(l\ell + j)q(l\ell + j) - \frac{p((l - \sigma)\ell + j)(\Delta_\ell \rho(l\ell + j))^2}{4\rho(l\ell + j)} \right] = \infty. \quad (22)$$

Then Theorem 2.1 and Corollary 1 in [9] are the same in the case when $\gamma = \beta = \ell = 1$. Also when $\sigma = 0$ and $p(k\ell + j) = 1$ and $\gamma = \beta = \ell = 1$ Theorem 2.1 and Corollary 3 in [10] are the same.

Theorem 2.4. Assume that (3) holds. Let $\rho(k\ell + j)$ be a real valued function. Furthermore, we assume that there exists a double fuction $H(m, k\ell + j)$; $m \geq k \geq 0$ such that (i) $H(m, m) = 0$ for $m \geq 0$, (ii) $H(m, k\ell + j) > 0$ for $m > k\ell + j \geq 0$, (iii) $\Delta_{2(\ell)} H(m, k\ell + j) = H(m, (k+1)\ell + j) - H(m, k\ell + j) \leq 0$ for $m \geq k\ell + j \geq 0$. If

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \frac{1}{H(m, 0)} \sum_{k=k_0}^{m-1} [H(m, k\ell + j)\rho(k\ell + j)q(k\ell + j) \\ & - \frac{(\rho((k+1)\ell + j))^2}{\rho(k\ell + j)} \left(h(m, k\ell + j) - \frac{\Delta_\ell \rho(k\ell + j)}{\rho((k+1)\ell + j)} \sqrt{H(m, k\ell + j)} \right)^2] = \infty, \end{aligned} \quad (23)$$

$$\text{where} \quad h(m, k\ell + j) = -\frac{\Delta_{2(\ell)} H(m, k\ell + j)}{\sqrt{H(m, k\ell + j)}}, m \geq k \geq 0,$$

$$\overline{\rho(k\ell + j)} = \frac{\beta \rho(k\ell + j) M^{(\gamma-1)/\gamma}}{(p((k - \sigma)\ell + j))^{1/\gamma} \eta^{1-\beta} ((k - \sigma + 1)\ell + j)^{1-\beta}}$$

Then every solution of equation (2) oscillates.

Proof. We proceed as in Theorem 2.1, we assume that equation (2) has a nonoscillatory solution, say $u((k - \sigma)\ell + j) > 0$ for all $k \geq k_0$. From the proof of Theorem 2.1, we obtain (18) for all $k \geq k_2$. From (18) we have for $k \geq k_2$,

$$\begin{aligned} \rho(k\ell + j)q(k\ell + j) & \leq -\Delta_\ell z(k\ell + j) + \frac{\Delta_\ell \rho(k\ell + j)}{\rho((k+1)\ell + j)} z((k+1)\ell + j) \\ & - \frac{\overline{\rho(k\ell + j)}}{(\rho((k+1)\ell + j))^2} w^2(k + \ell). \end{aligned} \quad (24)$$

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Therefore, we have

$$\begin{aligned} \sum_{k=n}^{m-1} H(m, k\ell + j) \rho(k\ell + j) q(k\ell + j) &\leq - \sum_{k=n}^{m-1} H(m, k\ell + j) \Delta_\ell z(k\ell + j) \\ &+ \sum_{k=n}^{m-1} \frac{H(m, k\ell + j) \Delta_\ell \rho(k\ell + j) z((k+1)\ell + j)}{\rho((k+1)\ell + j)} - \sum_{k=n}^{m-1} \frac{H(m, k\ell + j) \overline{\rho(k\ell + j)} z^2((k+1)\ell + j)}{(\rho((k+1)\ell + j))^2}. \end{aligned}$$

which yields, after summing by parts the first term in the right hand side.

$$\begin{aligned} &\sum_{k=n}^{m-1} H(m, k\ell + j) \rho(k\ell + j) q(k\ell + j) \\ &\leq H(m, k\ell + j) z(k\ell + j) + \sum_{k=n}^{m-1} z((k+1)\ell + j) \Delta_{2(\ell)} H(m, k\ell + j) + \sum_{k=n}^{m-1} H(m, k\ell + j) \\ &\quad \frac{\Delta_\ell \rho(k\ell + j)}{\rho((k+1)\ell + j)} z((k+1)\ell + j) - \sum_{k=n}^{m-1} H(m, k\ell + j) \frac{\overline{\rho(k\ell + j)}}{(\rho((k+1)\ell + j))^2} z^2((k+1)\ell + j) \\ &= H(m, k\ell + j) z(k\ell + j) - \sum_{k=n}^{m-1} h(m, k\ell + j) \sqrt{H(m, k\ell + j)} z((k+1)\ell + j) \\ &\quad + \sum_{k=n}^{m-1} H(m, k\ell + j) \frac{\Delta_\ell \rho(k\ell + j)}{\rho((k+1)\ell + j)} z((k+1)\ell + j) \\ &\quad - \sum_{k=n}^{m-1} H(m, k\ell + j) \frac{\overline{\rho(k\ell + j)}}{(\rho((k+1)\ell + j))^2} z^2((k+1)\ell + j) = H(m, k\ell + j) z(k\ell + j) \\ &\quad - \sum_{k=n}^{m-1} [z((k+1)\ell + j) \\ &\quad + \frac{\rho((k+1)\ell + j) \left(h(m, k\ell + j) \sqrt{H(m, k\ell + j)} - \frac{\Delta_\ell \rho(k\ell + j) H(m, k\ell + j)}{\rho((k+1)\ell + j)} \right)}{2\sqrt{H(m, k\ell + j) \rho(k\ell + j)}}] \Bigg]^2 \\ &\quad + \frac{1}{4} \sum_{k=n}^{m-1} \frac{(\rho((k+1)\ell + j))^2}{\rho(k\ell + j)} \left(h(m, k\ell + j) - \frac{\Delta_\ell \rho(k\ell + j)}{\rho((k+1)\ell + j)} \sqrt{H(m, k\ell + j)} \right)^2 \end{aligned}$$

Then,

$$\begin{aligned} & \sum_{k=k_1}^{m-1} [H(m, k\ell + j)\rho(k\ell + j)q(k\ell + j) \\ & \quad - \frac{(\rho((k+1)\ell + j))^2}{4\rho(k\ell + j)} \left(h(m, k\ell + j) - \frac{\Delta_\ell \rho(k\ell + j)\sqrt{H(m, k\ell + j)}}{\rho((k+1)\ell + j)} \right)^2] \\ & < H(m, k_1\ell + j)z(k_1\ell + j) \leq H(m, 0)z(k_1\ell + j) \end{aligned}$$

which implies that

$$\begin{aligned} & \sum_{k=0}^{m-1} [H(m, k\ell + j)\rho(k\ell + j)q(k\ell + j) \\ & \quad - \frac{(\rho((k+1)\ell + j))^2}{4\rho(k\ell + j)} \left(h(m, k\ell + j) - \frac{\Delta_\ell \rho(k\ell + j)\sqrt{H(m, k\ell + j)}}{\rho((k+1)\ell + j)} \right)^2] \\ & < H(m, 0) \left(z(k_1\ell + j) + \sum_{k=0}^{k_1-1} \rho(k\ell + j)q(k\ell + j) \right) \end{aligned}$$

Hence

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \frac{1}{H(m, 0)} \sum_{k=0}^{m-1} [H(m, k\ell + j)\rho(k\ell + j)q(k\ell + j) \\ & \quad - \frac{(\rho((k+1)\ell + j))^2}{4\rho(k\ell + j)} \left(h(m, k\ell + j) - \frac{\Delta_\ell \rho(k\ell + j)\sqrt{H(m, k\ell + j)}}{\rho((k+1)\ell + j)} \right)^2] \\ & < \left(z(k_1\ell + j) + \sum_{k=0}^{k_1-1} \rho(k\ell + j)q(k\ell + j) \right) < \infty. \end{aligned}$$

Corollary 2.5. Assume that all the assumptions of Theorem (2.4) hold, except the condition (23) is replaced by

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \frac{1}{m^\lambda} \sum_{k=0}^m \left[(m - (k\ell + j))^\lambda \rho(k\ell + j)q(k\ell + j) \right. \\ & \quad \left. - \frac{\lambda^2 (\rho((k+1)\ell + j))^2 (m - (k\ell + j))^{\lambda-2}}{4\rho(k\ell + j)} \right] = \infty. \end{aligned} \quad (25)$$

Then, every solution of equation (2) oscillates.

Corollary 2.6. Assume that all the assumptions of Theorem 2.4 hold, except the condition (23) is replaced by

$$\limsup_{m \rightarrow \infty} \frac{1}{(\log(m+1))^\lambda} \sum_{k=0}^{m-1} \left[\left(\log \frac{m+1}{(k+1)\ell+j} \right)^\lambda \rho(k\ell+j) q(k\ell+j) \right. \\ \left. \frac{(\rho((k+1)\ell+j))^2}{4\rho(k\ell+j)} \left(\frac{\lambda \left(\frac{m+1}{(k+1)\ell+j} \right)^{\frac{(\lambda-2)}{2}}}{(k+1)\ell+j} - \frac{\Delta_\ell \rho(k\ell+j) \sqrt{\left(\log \frac{m+1}{(k+1)\ell+j} \right)^\lambda}}{\rho((k+1)\ell+j)} \right)^2 \right] = \infty.$$

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