

AN ANALYSIS ON MARKOV RANDOM FIELDS (MRFs) USING CYCLE GRAPHS

F. Anitha Florence Vinola¹ and G. Padma²

^{1,2}Department of Mathematics

Sathyabama Institute of Science and Technology

Chennai-119, India.

E-mail: anithaflorence1981@gmail.com¹

E-mail: govindanpadma1970@gmail.com²

Abstract

An undirected graphical structure whose vertices are set of random variables having a Markov property is called a Markov Random Field (MRF). Some of the Markov properties are discussed with cycle graph and complete graph using undirected graphical representations. Belief propagation (Message passing algorithm) over the Markov random field has many useful applications, and has been successfully applied to several important computer vision problems. In coding theory, the error codes such as Low Density Parity Check (LDPC) codes and Turbo codes are mainly applied to minimize the errors, when the messages passed from one medium to another medium using belief propagation. This paper describes the relation between the belief propagation and maximal cliques in terms of undirected graphical structure.

AMS Subject Classification: 60J20, 60J05, 60J10, 60J25.

Key Words and Phrases: Markov random field, Belief propagation, maximal clique, undirected graph, error correcting codes.

1 Introduction

An undirected graphical structure with Markov properties is called a Markov Random Field (MRF). MRF has many useful applications in Bayesian networks, error correcting codes, wireless networks using Belief Propagation Algorithm (BPA). The importance of Belief propagation algorithm is to imply the marginal densities on every node of the graphical structure [1, 5, 7]. Hence the Belief propagation is also known as the message passing algorithm. BPA estimates the marginal densities for each unobserved nodes, conditional on any observed nodes. Belief propagation is mainly used in artificial intelligence and information theory [8, 9]. It explains the enormous applications in different fields such as low-density parity check codes, turbo codes etc.

Belief propagation was first introduced by Judea Pearl in 1982, who described the algorithm on trees and was later extended to poly trees. It seems to be powerful in many undirected graphical structures. BPA is generally presented as message update equations on a factor graphs involving messages between variable nodes and their neighbouring factor nodes and vice-versa [2, 4, 6]. The message transmitted from one medium to another medium in an undirected graphical structure is the way of generalising Belief propagation algorithm.

The variant of belief propagation algorithm is the Gaussian belief propagation when the underlying distributions are Gaussian [3]. The scope and features of this paper are arranged in the following way.

The definitions and graphical representations of factor graphs and theorems are discussed in chapter II, graphical representation of a Markov random field in chapter III, discussion of Markovian properties using cycle and complete graphs in chapter IV, belief propagation algorithm for coding and decoding messages using maximal cliques on undirected graphs in chapter V and conclusions in chapter VI.

2 Definitions and Theorems

Markov Random Field

A Markov random field or Markov network or undirected graphical model in which the vertices of a graph are the set of random variables having a Markov property described by an undirected graphs.

Markov random fields are undirected graphical structures that may be cyclic.

Clique

A clique is a sub graph of an undirected graphs such that every two distinct vertices in the clique are adjacent. That is its induced sub graph is complete.

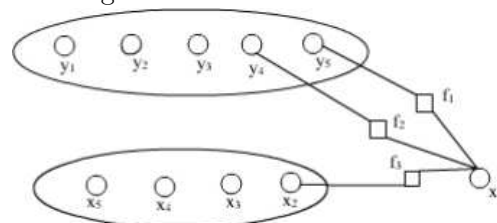
Maximal Clique

Maximal Clique is a clique that cannot include one more adjacent vertex to its vertex set, and it does not exist exclusively within the vertex set of a large clique.

Factor graph

A factor graph is a bipartite graph that refers to the factorization of a function. Factor graphs are used to represent the joint probability mass function of the variables and factorization of a probability distribution function that consist of the system in probability theory and its applications. A factor graph can be used to group the variable nodes and factor nodes, and gives the important information about statistical dependencies among these variables in probabilistic modelling of systems. The decoding of capacity-approaching error correcting codes, such as LDPC and turbo codes is the most powerful success of factor graphs and the sum-product algorithm.

In a factor graph circles are represented as variable nodes, square boxes are represented as factors and the straight lines are represented as an edge between the variables and the factors.



Theorem 1. An irreducible Markov chain is transient for undirected graphical structure iff for some state i there exists a non zero vector y_j such that $\sum p_{ij}y_j$ for all $j \neq i$ and $|y_j| < 1$ for all j (or) the graph should contain the maximal clique of vertex which is less than or equal to half of the number of vertices of a given graph.

3 Graphical Representation of a Markov Random Field

The following undirected graphical structure is the representation of Markov random field.

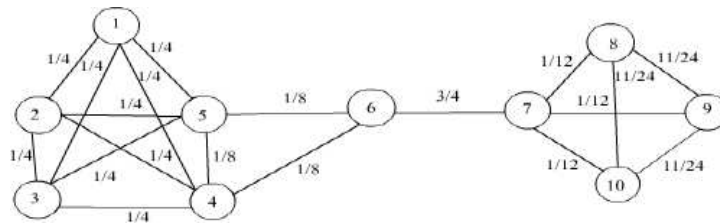


Figure 1: (Undirected graphical structure)

The transition probability matrix (tpm) corresponding to the undirected graphical structure is given by

$$P = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & 0 & \frac{11}{24} & \frac{11}{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & \frac{11}{24} & 0 & \frac{11}{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{12} & \frac{11}{24} & \frac{11}{24} & 0 \end{pmatrix}$$

For the above tpm, the zero sub-square matrix of order $4 < 5 = \frac{10}{2} = \frac{n}{2}$, the chain is irreducible [10] and using Theorem 1 the chain is transient.

4 Discussion of Markovian Properties for Cycle and Complete Graphs

A graph in which each distinct pair of vertices are adjacent is called a complete graph. If the degree of all the vertices of a graph are equal (n degrees), then the graph is called n -regular graph.

A graph in which the starting and ending vertices are same is called a cycle graph. Consider the following 5-regular complete graph.

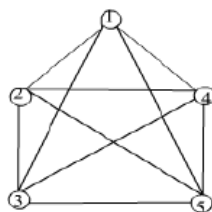


Figure 2: (complete graph)

The tpm corresponding to the above undirected complete graph is

$$P = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

For a 4-regular complete graph (Figure 2), only the diagonal elements of a tpm are zero. Since the complete graph is always an irreducible Markov chain [10] and the states are aperiodic. Since the states are finite and irreducible, they are non-null persistent which gives the result that the complete graph is always ergodic. Consider the following cycle graph.

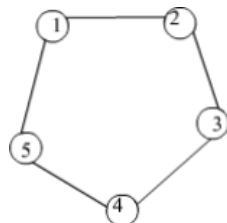


Figure 3: (Cycle graph)

The tpm corresponding to the above undirected cycle graph is

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

For a cycle graph, the tpm can be constructed and it is found that the probability of all the states in any one step becomes non-zero. (i.e.) $p_{ij}^n > 0$. Therefore the chain is irreducible and the states are finite, the chain is non-null persistent.

If the given undirected graphical structure is a cycle of odd vertices, the chain is aperiodic.

If the given undirected graphical structure is a cycle of even vertices, the chain is periodic of period 2.

A cycle graph also possesses the nature of a random walk, which is a mathematical formalization of a path that consists of a succession of random steps in Markov random field.

5 Belief Propagation Algorithm in Terms of Maximal Cliques

There are different ways of defining and tracking the set of regions in a graph that can exchange messages. One method uses ideas introduced by Kikuchi in the physics literature, and is known as cluster variation method. The two different improvements

to belief propagation are the cluster variation method and the survey propagation algorithms. Belief propagation algorithm is used for tracking, partition and many image representation tasks. Belief propagation is a probabilistic graphical model which gives the detailed knowledge of probability distributions that shares a common structure. Hammersly-Clifford theorem helps to identify the exact structure of the graph which is nothing but the product of all the maximal cliques of the graph.

The probability of an image x , which are considered to be unobserved nodes under a Markov random field can be written as a product of all the maximal cliques that are the observed nodes.

$$p(x) = \frac{1}{Z} \prod_C \psi(X_C), \text{ where } X_C \text{ is the image region correspond-}$$

ing to the clique C , ψ is a potential function of the clique, and Z is a normalization function which makes the total probability under integral area is equal to one.

In belief propagation, for finding the marginal probability at every node, a message is a re-usable partial sum for the marginalization calculations.

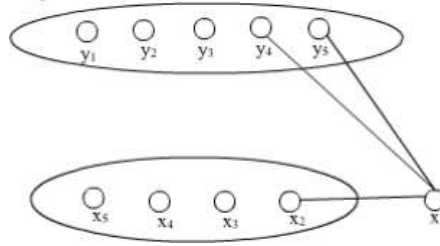


Figure 4:

The marginal probability of the image node x_1 with respect to the observed nodes y_i 's gives

$$\begin{aligned} p\left(\frac{x_1}{\bar{y}}\right) &= \frac{1}{p(\bar{y})} \sum_{x_2} \sum_{x_3} \sum_{x_4} \sum_{x_5} \phi_{12}(x_1, x_2) \phi_{13}(x_1, x_3) \phi_{14}(x_1, x_4) \\ &\quad \times \phi_{15}(x_1, x_5) \psi_{41}(y_4, x_1) \psi_{51}(y_5, x_1) \\ &= \frac{1}{p(\bar{y})} m_{41}(x_1) m_{51}(x_1). \end{aligned}$$

Similarly the marginal probabilities of the remaining nodes gives the same result that the Figure 4 shows the interrelations among the observed nodes (y_i 's) and image nodes (x_i 's). Since nodes of the maximal clique are adjacent, when the message passed on to the observed nodes, belief propagation applied on the maximal cliques minimizes the run time error.

6 Conclusion

Graphs are an interesting and exciting way of representing and picturing the relationship between many variables. A graph helps us to identify the conditional independence relationship between the variables. In this paper, the belief propagation algorithm is graphically discussed in terms of maximal cliques. As the belief propagation algorithm is used in LDPC and Turbo codes, to minimize the error is discussed in detail in terms of undirected graphical structure. The given graphical circuit contains a maximal clique of size greater than or equal to, half the size of the graph. Therefore, Markov random field along with the maximal cliques improvise the error correction of LDPC codes and Turbo codes.

References

- [1] S. Benedetto, G. Montorsi, D. Divsalar, F. Pollara, Soft-output decoding algorithms in iterative decoding of Turbo codes, *Technical Report*, 42-124, JPL TDA (1996).
- [2] F.R. Kschischang, B.J. Frey, H.A. Loeliger, Factor graphs and the sum-product algorithm, *IEEE Transactions on Information Theory* (1998).
- [3] M.I. Jordan, Z. Ghahramani, T.S. Jakkola, L.K. Saul, An introduction to variational methods for graphical methods, *Machine Learning*, **37** (1999), 183-233.
- [4] F.R. Kschischang, B.T. Frey, H.A. Loelinger, Factor graphs and sum product algorithm, *IEEE T. Info Th.*, **47**(2) (2001), 498-519.

- [5] M.F. Tappen, W.T. Freeman, Comparison of graph cuts with belief propagation for stereo, using identical MRF parameters, *ICCV*, **2** (2003), 900-907.
- [6] Lecture Notes: Factor graphs and belief propagation Marc Toussaint Machine Learning & Robotics group, TU Berlin Franklinstr. 28/29, FR 6-9, 10587 Berlin, Germany March 14, 2008.
- [7] T.S. Yedidia, W.T. Freeman, Y. Weiss, Bethe free energies, Kikuchi approximations, and belief propagation algorithms, *MERL Technical Report* 2001-16 (2001).
- [8] G. Padma, C. Vijayalakshmi, A Comparison on Soft-Error Correcting Codes of memory Cells in a Markov Random field, *Proceedings of the international conference on cloud computing and eGovernance* (2012), 59-64, ISBN: 978-81-920575-0-7.
- [9] G. Padma, C. Vijayalakshmi, Implementation of Belief propagation Iterative Method on Markov chains by Designing Bayesian Networks, *CiiT International Journal of Artificial Intelligent Systems and Machine Learning*, **3**(6) (2011).
- [10] F. Anitha Florence Vinola, G. Padma, An analysis on the Markov chain properties using pictorial representation, *International Conference on Innovations in information Embedded and Communication Systems (ICIIECS)*, **III** (2017), 560–564, ISBN: 978-1-5090-3293-8.

