Properties of $\Delta^*$-Locally Closed Sets
In Topological Spaces

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Abstract

The objective of this paper is to analyze the properties of $\Delta^*$-locally closed sets in topological spaces. Further as an application of $\Delta^*$-locally closed sets, two spaces named as $\Delta^*$-door space and $\Delta^*$-submaximal spaces are also introduced and investigated in this paper.

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1 Introduction

The study of locally closed sets was initiated by Bourbaki [1] in the year 1966. The definition and the notation of locally closed sets were established by M.Ganster et.al., [6] in 1989. This idea was
extended to generalised closed sets and named as generalised locally closed sets by K.Balachandran.et.al.,[2] in 1966. Since then several significant concepts based on locally closed sets have been done by many researchers. In this paper, three types of locally closed sets namely, $\Delta^*lc$-sets, $\Delta^*lc^*$-sets and $\Delta^*lc^{**}$-sets and two new spaces namely, $\Delta^*$-Door space and $\Delta^*$-submaximal space using $\Delta^*$-closed sets were introduced and discussed their nature in topological spaces. Further we proved some applications of these new class of locally closed sets using separation axioms of $\Delta^*$-closed sets. Throughout this paper $(X, \tau)$ denotes a topological space on which no separation axioms are stated unless or otherwise specified. 

**Remark**: In 2014, a new class of closed sets namely, $\Delta^*$-closed sets [8] were introduced and initially denoted by $\delta(\delta g)^*$-closed sets by the authors.

## 2 Preliminaries

**Definition 1.** Let $M$ be a subset of a topological space $(X, \tau)$.

i) If $M = \text{int}(\text{cl}(M))$ then $M$ is called a regular open set.[15]

ii) If $M$ is the union of regular open sets then $M$ is called a $\delta$-open set.[14]

iii) If $\text{cl}(M) \subseteq R$ whenever $M \subseteq R$ where $R$ is open in $(X, \tau)$ then $M$ is called a generalised closed set (denoted as $g$-closed set) in $(X, \tau)$. [7]

iv) If $\delta \text{cl}(M) \subseteq R$ whenever $M \subseteq R$ where $R$ is open in $(X, \tau)$ then $M$ is called a $\delta$-generalised closed set (denoted as $\delta g$-closed set) in $(X, \tau)$. [3]

v) If $\text{cl}(M) \subseteq R$ whenever $M \subseteq R$ where $R$ is $\delta$-open in $(X, \tau)$ then $M$ is called a generalised $\delta$-closed set (denoted as $g\delta$-closed set) in $(X, \tau)$. [4]

vi) If $\text{cl}(M) \subseteq R$ whenever $M \subseteq R$ where $R$ is $\pi$-open in $(X, \tau)$ then $M$ is called a $\pi$-generalised closed set (denoted as $\pi g$-closed set) in $(X, \tau)$.
vii) If \( \delta cl(M) \subseteq R \) whenever \( M \subseteq R \) where \( R \) is \( \delta g \)-open in \((X, \tau)\) then \( M \) is called a \( \Delta^* \)-closed set in \((X, \tau)\).

**Definition 2.** Let \( M \) be a subset of a topological space \((X, \tau)\). Then the \( \Delta^* \)-closure operator \([9]\) of \( M \) in \((X, \tau)\) is denoted by \( \Delta^* cl(M) \) and defined as below.

\[
\Delta^* cl(M) = \bigcap \{ S \subseteq X : M \subseteq S \text{ and } S \text{ is } \Delta^* \text{-closed in } (X, \tau) \}.
\]

**Definition 3.** If every \( g \delta \)-closed subset of \((X, \tau)\) is \( \Delta^* \)-closed in \((X, \tau)\) then \((X, \tau)\) is said to be a \( gT_{\Delta^*} \)-space\([10]\).

**Definition 4.** A map \( f : (X, \tau) \rightarrow (Y, \delta) \) is called \( \Delta^* \)-irresolute if \( f^{-1}(G) \) is a \( \Delta^* \)-open set in \((X, \tau)\) for every \( \Delta^* \)-open set \( G \) in \((Y, \sigma)\)\([12]\).

**Remark 1.** If \( M \) is \( \delta g \)-open as well as \( \Delta^* \)-closed set of \((X, \tau)\) then \( M \) is a \( \delta \)-closed set of \((X, \tau)\)\([13]\).

**Remark 2.** The intersection of two \( \Delta^* \)-open sets is a \( \Delta^* \)-open set\([13]\).

**Remark 3.** Every \( \Delta^* \)-open set is a \( \pi g \)-open set\([13]\).

### 3 Properties of \( \Delta^* \)-Locally Closed Sets

**Definition 5.** Consider a subset \( M \) of \((X, \tau)\). Then it is said to be

a) \( \Delta^* \)-locally closed set, i.e., \( \Delta^* lc \)-set if there exists a \( \Delta^* \)-open set \( R \) and a \( \Delta^* \)-closed set \( S \) of \((X, \tau)\) such that \( M = R \cap S \).

b) \( \Delta^* lc^* \)-set if there exists a \( \Delta^* \)-open set \( R \) and a \( \delta \)-closed set \( S \) of \((X, \tau)\) such that \( M = R \cap S \).
c) $\Delta^*lc^{**}$-set if there exists a $\delta$-open set $R$ and a $\Delta^*$-closed set $S$ of $(X, \tau)$ such that $M = R \cap S$.

The collection of all $\Delta^*lc$ (resp., $\Delta^*lc^*$-sets, $\Delta^*lc^{**}$) sets of $(X, \tau)$ is represented by $\Delta^*LC(X, \tau)$ (respectively $\Delta^*LC^*(X, \tau)$, $\Delta^*LC^{**}(X, \tau)$).

**Proposition 1.** In $(X, \tau)$ the following inclusions are proved.

a) $\delta LC(X, \tau) \subseteq \Delta^*LC(X, \tau)$.

b) $\delta LC(X, \tau) \subseteq \Delta^*LC^*(X, \tau) \subseteq \Delta^*LC(X, \tau)$.

c) $\delta LC(X, \tau) \subseteq \Delta^*LC^{**}(X, \tau) \subseteq \Delta^*LC(X, \tau)$.

**Proof.** Using the result that every $\delta$-closed set is $\Delta^*$-closed set in $(X, \tau)$. [8], the proof can be observed.

**Remark 4.** The reverse implications of the above results are not true as seen from the following example.

**Counter Example 1.** Let $X = \{t_1, t_2, t_3\}$ and $\tau = \{\phi, X, \{t_1\}\}$. Then $\delta LCX, \tau) = \{\phi, X\}$; $\Delta^*LC(X, \tau) = \{\phi, X, \{t_1\}, \{t_2, t_3\}\}$; $\Delta^*LC^*(X, \tau) = \{\phi, X, \{t_1\}\}$; $\Delta^*LC^{**}(X, \tau) = \{\phi, X, \{t_2, t_3\}\}$.

**Remark 5.** It can be viewed by the above example that $\Delta^*LC^*(X, \tau)$ and $\Delta^*LC^{**}(X, \tau)$ are independent.

**Proposition 2.** If $\Delta^*C(X, \tau) \subseteq \delta GO(X, \tau)$ then $\Delta^*LC^{**}(X, \tau) = \delta LC(X, \tau)$.

**Proof.** By Proposition 1(b), $\delta LC(X, \tau) \subseteq \Delta^*LC^{**}(X, \tau)$—(1). Let $M$ be a $\Delta^*lc^{**}$-set. Then there exists a $\delta$-open set $R$ and a $\Delta^*$-closed set $S$ of $X$ such that $M = R \cap S$. By the assumption $S$ is $\delta g$-open. Therefore $S$ is $\delta$-closed set of $X$. [Remark 1]. Thus $M$ is $\delta lc$-set and $\Delta^*LC^{**}(X, \tau) \subseteq \delta LC(X, \tau)$ ———(2). Hence from (1) and (2), $\Delta^*LC^{**}(X, \tau) = \delta LC(X, \tau)$. 

\[\square\]
Proposition 3. If $M$ is a subset of $(X, \tau)$ and if $M \in \Delta^* LC(X, \tau)$ then $M = R \cap \Delta^* cl(M)$ for some $\Delta^*$-open set $R$ in $(X, \tau)$.

Proof. Let $M \in \Delta^* LC(X, \tau)$. Then there exists a $\Delta^*$-open set $R$ and a $\Delta^*$-closed set $S$ of $X$ such that $M = R \cap S$. Since $M \subseteq R$ and $M \subseteq \Delta^* cl(M)$, $M \subseteq R \cap \Delta^* cl(M)$ ——– (1). Conversely by the definition of $\Delta^*$-closure, $\Delta^* cl(M) \subseteq S$ and hence $R \cap \Delta^* cl(M) \subseteq R \cap S = M$ ——– (2). Therefore from (1) and (2), $M = R \cap \Delta^* cl(M)$.

Proposition 4. For a subset $M$ of $(X, \tau)$, if $M \in \Delta^* LC^{**}(X, \tau)$ then $M = R \cap \Delta^* cl(M)$ for some $\delta$-open set $R$ in $(X, \tau)$.

Proof. Let $M \in \Delta^{**} LC(X, \tau)$. Then by the definition, $M = R \cap S$ where $R$ is a $\delta$-open set and $S$ is a $\Delta^*$-closed set containing $M$. Since $S$ is a $\Delta^*$-closed set, $\Delta^* cl(M) \subseteq S$ which implies that $R \cap \Delta^* cl(M) \subseteq R \cap S = M$. Since $M \subseteq R$ and $M \subseteq \Delta^* cl(M)$, $M \subseteq R \cap \Delta^* cl(M)$. Therefore $M = R \cap \Delta^* cl(M)$ where $R$ is a $\delta$-open set in $(X, \tau)$.

Proposition 5. The following results are true for any two subsets $M$ and $N$ of $(X, \tau)$.

a) If $M, N \in \Delta^* LC^*(X, \tau)$, then $M \cap N \in \Delta^* LC^*(X, \tau)$.

b) If $M \in \Delta^* LC(X, \tau)$ and $N$ is $\Delta^*$-open then $M \cap N \in \Delta^* LC(X, \tau)$.

c) If $M \in \Delta^* LC^{*}(X, \tau)$ and $N$ is $\Delta^*$-open then $M \cap N \in \Delta^* LC^{*}(X, \tau)$.

d) If $M \in \Delta^* LC^{**}(X, \tau)$ and $N$ is $\Delta^*$-open then $M \cap N \in \Delta^* LC^{**}(X, \tau)$.

Proof. a) Follows from the fact that the intersection of two $\Delta^*$-open sets is $\Delta^*$-open and by the Remark 2.

b) and c) Follows from the fact that the intersection of two $\Delta^*$-open sets is $\Delta^*$-open.

d) Follows from the fact that the intersection of $\delta$-open and $\Delta^*$-open sets is $\Delta^*$-open.[13].
4 Applications of \( \Delta^* \)-Locally Closed Sets

**Definition 6.** A subset \( M \) of \( (X,\tau) \) is called a \( \Delta^* \)-dense set if \( \Delta^*\text{cl}(M) = X \).

**Example 1.** Let \( X = \{t_1, t_2, t_3\} \) and \( \tau = \{\phi, X, \{t_1\}, \{t_1, t_2\}\} \). Then the \( \Delta^* \)-dense sets are \( X \) and \( \{t_1, t_2\} \).

**Proposition 6.** In a topological space \( (X,\tau) \), every \( \Delta^* \)-dense set is a \( \delta \)-dense set but the converse is not true.

**Proof.** Let \( M \) be a \( \Delta^* \)-dense set in \( (X,\tau) \). Then \( \Delta^*\text{cl}(M) = X \).

Since \( \Delta^*\text{cl}(M) \subseteq \delta\text{cl}(M) \), \( \delta\text{cl}(M) = X \). Hence \( M \) is \( \delta \)-dense. \( \Box \)

**Counter Example 2.** Let \( X = \{t_1, t_2, t_3\} \) and \( \tau = \{\phi, X, \{t_1\}, \{t_1, t_2\}\} \). Then the subset \( \{t_3\} \) is \( \delta \)-dense in \( (X,\tau) \) but not \( \Delta^* \)-dense set in \( (X,\tau) \) since \( \Delta^*\text{cl}\{t_3\} = \{t_3\} \neq X \) whereas \( \Delta^*\text{cl}\{t_1, t_2\} = X \).

**Proposition 7.** In a topological space \( (X,\tau) \), every \( \pi g \)-dense set is \( \Delta^* \)-dense set but not conversely.

**Proof.** Let \( M \) be a \( \pi g \)-dense set in \( X \). Then \( \pi g\text{cl}(M) = X \).

Since \( \pi g\text{cl}(M) \subseteq \Delta^*\text{cl}(M) \), \( \Delta^*\text{cl}(M) = X \). Hence \( M \) is \( \Delta^* \)-dense. \( \Box \)

**Counter Example 3.** Let \( X = \{t_1, t_2, t_3\} \) and \( \tau = \{\phi, X, \{t_1\}\} \).

Then the subset \( \{t_1, t_2\} \) is \( \Delta^* \)-dense but not \( \pi g \)-dense since \( \pi g\text{cl}\{t_1, t_2\} = \{t_1\} \neq X \).

**Definition 7.** A topological space \( (X,\tau) \) is said to be a \( \Delta^* \)-door space if each subset of \( (X,\tau) \) is either \( \Delta^* \)-open or \( \Delta^* \)-closed in \( (X,\tau) \).
Example 2. Let $X = \{t_1, t_2, t_3\}$ and $\mathcal{V} = \{\phi, X, \{t_1\}, \{t_1, t_2\}\}$. Then $(X, \tau)$ is a $\Delta^*$-door space.

Remark 6. If $(X, \tau)$ is a $\Delta^*$-door space then $\Delta^*LC(X, \tau) = P(X)$.

Definition 8. A topological space $(X, \tau)$ is said to be a $\Delta^*$-submaximal (resp., $\Delta^{**}$-submaximal) space if every $\Delta^*$-dense (resp., $\delta$-dense) subset is $\Delta^*$-open in $(X, \tau)$.

Proposition 8. Every $\Delta^*$-submaximal space is a $\pi g$-submaximal space but not conversely.

Proof. Let $(X, \tau)$ be a $\Delta^*$-submaximal space and $M$ is $\pi g$-dense subset of $(X, \tau)$. Since every $\pi g$-dense subset is $\Delta^*$-dense (Proposition 7), $M$ is $\Delta^*$-dense and $M$ is $\Delta^*$-open. Therefore $M$ is $\pi g$-open. [Remark 3]. Hence $(X, \tau)$ is a $\pi g$-submaximal space.

Counter Example 4. Let $X = \{t_1, t_2, t_3\}$ and $\tau = \{\phi, X, \{t_1\}\}$. Then $(X, \tau)$ is $\pi g$-submaximal space but not a $\Delta^*$-submaximal space since the subset $\{t_1, t_2\} \in (X, \tau)$ is $\Delta^*$-dense but not $\Delta^*$-open.

Proposition 9. Every $\Delta^{**}$-submaximal space is a $\Delta^*$-submaximal space but not conversely.

Proof. Let $(X, \tau)$ be a $\Delta^{**}$-submaximal space and $M$ be a $\Delta^*$-dense subset of $(X, \tau)$. By Proposition 6, $M$ is $\delta$-dense in $(X, \tau)$. By the assumption, $M$ is $\Delta^*$-open and hence $(X, \tau)$ is a $\Delta^*$-submaximal space.

Counter Example 5. Let $X = \{t_1, t_2, t_3\}$ and $\mathcal{V} = \{\phi, X, \{t_1, t_2\}\}$. Then $(X, \tau)$ is a $\Delta^*$-submaximal as the $\Delta^*$-dense subsets $X$ and $\{t_1, t_2\}$ are $\Delta^*$-open. But $(X, \tau)$ is not a $\Delta^{**}$-submaximal since the
subset \{t_1, t_2\} in (X, \tau) is \(\delta\)-dense but not \(\Delta^*\)-open in \((X, \tau)\).

**Proposition 10.** Let \(f : (X, \tau) \to (Y, \sigma)\) be a \(\Delta^*\)-irresolute map. Then the following statements are true.

a) If \(N \in \Delta^*LC(Y, \sigma)\) then \(f^{-1}(N) \in \Delta^*LC(X, \tau)\).

b) If \(N \in \Delta^*LC(Y, \sigma)\) then \(f^{-1}(N) \in g\delta LC(X, \tau)\).

c) If \(N \in g\delta LC(Y, \sigma)\) and \((Y, \sigma)\) is a \(g\delta T_{\Delta^*}\)-space then \(f^{-1}(N) \in \Delta^*LC(X, \tau)\).

**Proof.** Let \(f : (X, \tau) \to (Y, \sigma)\) be a \(\Delta^*\)-irresolute map.

a) \(N \in \Delta^*LC(Y, \sigma)\). Then there exists a \(\Delta^*\)-open set \(G\) and \(\Delta^*\)-closed set \(H\) such that \(N = G \cap H\) which implies that \(f^{-1}(N) = f^{-1}(G) \cap f^{-1}(H)\). Since \(f\) is \(\Delta^*\)-irresolute, \(f^{-1}(G)\) and \(f^{-1}(H)\) are \(\Delta^*\)-open and \(\Delta^*\)-closed respectively. Hence \(f^{-1}(N) \in \Delta^*LC(X, \tau)\).

b) Let \(N \in \Delta^*LC(Y, \sigma)\). Then there exists a \(\Delta^*\)-open set \(G\) and \(\Delta^*\)-closed set \(H\) such that \(N = G \cap H\) which implies that \(f^{-1}(N) = f^{-1}(G) \cap f^{-1}(H)\). Since \(f\) is \(\Delta^*\)-irresolute, \(f^{-1}(G)\) and \(f^{-1}(H)\) are \(\Delta^*\)-open and \(\Delta^*\)-closed respectively. Since every \(\Delta^*\)-closed is \(g\delta\)-closed [8], \(f^{-1}(G)\) and \(f^{-1}(H)\) are \(g\delta\)-open and \(g\delta\)-closed respectively. Therefore \(f^{-1}(N) \in g\delta LC(X, \tau)\).

c) Let \(N \in g\delta LC(Y, \sigma)\). Then there exists a \(g\delta\)-open set \(G\) and a \(g\delta\)-closed set \(H\) in \((Y, \sigma)\) such that \(N = G \cap H\). Since \((Y, \sigma)\) is a \(g\delta T_{\Delta^*}\)-space, \(G\) is \(\Delta^*\)-open and \(\Delta^*\)-closed also. Then \(N \in \Delta^*LC(Y, \sigma)\). Hence by the result (a), \(f^{-1}(N) \in \Delta^*LC(X, \tau)\).

\(\square\)

**References**


