Operation- contra-continuous functions and Hausdorff Spaces with $\alpha-\gamma$-I-open sets in ideal topological spaces

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Abstract

In this paper the concept of contra-$\alpha-\gamma$-I-continuous functions is introduced and some of the properties are analyzed. Further the concept of $\alpha-\gamma$-I-Hausdorff Spaces is introduced and studied.

Key words: contra-$\alpha-\gamma$-I-continuous function, $\alpha-\gamma$-I-Hausdorff Spaces
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1 Introduction

O.Njastad$^1$ introduced $\alpha$-open sets in a topological space and studied some of their properties. The concept of semiopen sets , preopen sets and semi-preopen sets were introduced respectively by N.Levine$^2$, HH. Corson and E.Michael$^4$ and D.Andrijevic$^4$. The
term “preopen” was introduced by AS. Mashhour et al[5] and studied some of its basic properties. D. Andrijevic[6] introduced a new class of topology generated by preopen sets and the corresponding closure and interior operators. S. Kasahara [7] defined the concept of an operation on topological spaces. H. Ogata[8] called the operation $\alpha$ as $\gamma$ operation and introduced the notion of $\tau_\gamma$ which is the collection of all $\gamma$-open sets in a topological space $(X, \tau)$.

N. Kalaivani and G. Sai Sundara Krishnan[9] introduced $\alpha$-$\gamma$-open sets in a topological space $(X, \tau)$ and introduced the notion of $\tau_{\alpha\gamma}$ which is the collection of all $\alpha$-$\gamma$-open sets in a topological space. N. Kalaivani et al [10] introduced $\alpha$-$\gamma$-I-open sets in a topological space $(X, \tau, I)$ and introduced the notion of $\tau_{\alpha\gamma I}$ which is the collection of all $\alpha$-$\gamma$-I-open sets in a topological space.

[11] One historical line of investigation which led to the study of ideals was motivated by the desire to generalize the concepts of “limit point”, “closure point” and “condensation point”. Another historical line of investigation which naturally involves the concept of an ideal is motivated by the observation that many interesting theorems concern description of global properties of sets or functions from their local properties.

2 Motivation for the introduction of $\alpha$-$\gamma$-I-open sets, contra-$\alpha$-$\gamma$-I-continuous functions and $\alpha$-$\gamma$-I-Hausdorff space

By the introduction of $\alpha$-$\gamma$-I-open sets and contra-$\alpha$-$\gamma$-I-continuous functions using the concept of $\alpha$-$\gamma$-open sets we try to unify the topological ideal theory and operation theory in a topological space. The Kuratowski* unifies the concept of local functions in topological ideal theory and the concept of operation-closures in operation theory.

In this paper in section 4 the notion of contra-$\alpha$-$\gamma$-I-continuous functions is introduced and some of their properties are analyzed.

In section 4 the concept of $\alpha$-$\gamma$-I-Hausdorff spaces is introduced and studied their properties.

3 Preliminaries

In this section some of the basic Definitions and Theorems are recalled.

Let $(X, \tau)$ be a topological space and $I$ be an ideal of subsets of $X$. An ideal is defined as a nonempty collection $I$ of subsets of $X$ satisfying the following two conditions: (i) If $A \in I$ and $B \subset A$ then $B \in I$ (heredity); (ii) If $A \in I$ and $B \in I$, then $A \cup B \in I$ (finite additivity). An ideal topological space is a topological space $(X, \tau)$ with an ideal $I$ on $X$ and is denoted by $(X, \tau, I)$. For a subset $A \subset X$, $A^*(I) = \{x \in X : U \cap A \notin I$ for each neighbourhood $U$ of $x\}$ is called the local function of $A$ with respect to $I$ and $\tau[12]$. We write $A^*$ instead of $A^*(\tau, I)$. $X^*$ is often a proper subset of $X$. The hypothesis $X = X^*$ [14] is equivalent to the hypothesis $\tau \cap I = \phi[14]$. For every ideal topological space $(X, \tau, I)$, there exists a topology $\tau^*(I)$, finer
than τ, generated by β(I, τ) = \{U | I \in \tau \text{ and } I \in I\}, but in general β(I, τ) is not always a topology[15]. Additionally, cl*(A) = A∪A* defines a Kuratowski[12] closure operator for τ*(I).

Throughout this paper (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of (X, τ), cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A respectively. We define Kuratowski* closure operator as τγ-cl*(A) = A∪A*, where is the local function of A with respect to I, τ and γ.

The collection of complements of sets in a proper ideal will be a nonempty collection of nonempty sets closed under the operations of superset and finite intersection. Such a collection is called a filter; hence an ideal is some times called a dual filter.

An operation γ[8] on the topology τ on a given topological space (X, τ) is a function from the topology itself into the power set P(X) of X such that V ⊆ V' for each V ∈ τ. The γ-operation is called an expansion or magnification.

Let (X, τ) be a topological space and A be a subset of X. Then A is said to be (i) α-open set[1] if A ⊆ int(cl(int(A))). (ii) semi-open set [2] if A ⊆ cl(int(A)). (iii) pre-open set[4] if A ⊆ int(cl(A)). (iv) semi-preopen set[5] if A ⊆ cl(int(cl(A))).

A subset A is said to be: (i) a γ-open set[8] if for each x ∈ A there exists an open set U such that x ∈ U and U ⊆ A. τγ denotes the set of all γ-open sets in (X, τ). (ii) γ-semi-open set [14] if and only if A ⊆ τγ-cl(τγ-int(A)). (iii) γ-preopen set[15] if and only if A ⊆ τγ-int(τγ-cl(A)). (iv) γ-semi-preopen set [15] if and only if A ⊆ τγ-cl(τγ-int(τγ-cl(A))).

(i) The τγ-interior of A[14] is defined as the union of all γ-open sets contained in A and it is denoted τγ-int(A). τγ-int(A) = ∪ \{U : U is a γ-open set and U ⊆ A\}. (ii) The τγ-closure of A[9] is defined as the intersection of all γ-closed sets containing A and it is denoted by τγ-cl(A). τγ-cl(A) = \bigcap \{F : F is a γ-closed set and A ⊆ F\}. A subset A of X is said to be a α-γ-open set if and only if A ⊆ τγ-int(τγ-cl(τγ-int(A))).

Let (X, τ, I) be an ideal topological space and γ be an operation on τ. Then a subset A of X is said to be a α-γ-I-open set[10] if and only if A ⊆ τγ-int(τγ-cl*(τγ-int(A))). A subset A of X is said to be an α-γ-I-closed set[10] if and only if X−A is α-γ-I-open, which is equivalently, A is an α-γ-I-closed set if and only if A ⊆ τγ-cl(τγ-int*(τγ-cl(A))). The τα−γ-I-interior of A[10] is the union of all α-γ-I-open sets contained in A and it is denoted by τα−γ-I-int(A). τα−γ-I-int(A) = \bigcup \{U : U is a α-γ-I-open set and U ⊆ A\}. The τα−γ-I-closure of A[10] is the intersection of all α-γ-I-closed sets containing A and it is denoted by τα−γ-I-cl(A). τα−γ-I-cl(A) = \bigcap \{F : F is a α-γ-I-closed set and A ⊆ F\}.
A function \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is said to be \( \alpha-\gamma-I \)-continuous function \([10]\) if for every \( V \in \sigma \), \( f^{-1}(V) \) is an \( \alpha-\gamma \)-open set in \((X, \tau, I)\). A space \((X, \tau)\) is called a \( \alpha-\gamma \)-Hausdorff space\([9]\) if for each distinct points \( x, y \in X \) there exists \( \alpha-\gamma \)-open sets \( U, V \) such that \( x \in U, y \in V \) and \( U \cap V = \emptyset \). A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be \( \alpha-\gamma \)-irresolute function \([18]\) if and only if for any \( \alpha-\gamma \)-open subset \( V \) of \( Y \), \( f^{-1}(V) \) is an \( \alpha-\gamma \)-open set in \( X \). A function \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \) is said to be an \( \alpha-\gamma-I \)-irresolute function if for \( V \in \tau_{\alpha-\gamma-J} \), \( f^{-1}(V) \in \tau_{\alpha-\gamma-I} \).

Let \((X, \tau)\) be a topological space and \( A \) be a subset of \( X \). The set \( \cap \{U \in \tau : A \subseteq U\} \) is called the kernel of \( A \), and is denoted by \( \ker(A)\)[20]. Let \( A \) and \( B \) be subsets of \( X \), then \([21]\)

(i) \( x \in \ker(A) \) if and only if \( A \cap F \neq \emptyset \) for any \( \alpha-\gamma \)-closed set \( F \) that contains \( x \).

(ii) \( A \subseteq \ker(A) \) and \( A = \ker(A) \) if \( A \) is \( \alpha-\gamma \)-open.

(iii) If \( A \subseteq B \), then \( \ker(A) \subseteq \ker(B) \).

**Theorem 4.1** Let \((X, \tau)\) be a topological space, \( A \) and \( B \) be subsets of \( X \), then

\[
\{i\} \quad x \in \tau_{\alpha-\gamma} \ker(A) \text{ if and only if } A \cap F \neq \emptyset \text{ for any } \alpha-\gamma \text{-closed set } F \text{ that contains } x.
\]

\[
\{ii\} \quad A \subseteq \tau_{\alpha-\gamma} \ker(A) \text{ and } A = \tau_{\alpha-\gamma} \ker(A) \text{ if } A \text{ is } \alpha-\gamma \text{-open}.
\]

\[
\{iii\} \quad \text{If } A \subseteq B, \text{ then } \tau_{\alpha-\gamma} \ker(A) \subseteq \tau_{\alpha-\gamma} \ker(B).
\]

**Proof** The proof follows from the Definition 4.1.

**Definition 4.2** A topological space \((X, \tau)\) is said to be an \( \alpha-\gamma-R_0 \) space if for each \( \alpha-\gamma \)-open set \( U, x \in U \) implies that \( \tau_{\alpha-\gamma} \ker(\{x\}) \subseteq U \).

**Theorem 4.2** Let \((X, \tau)\) be a topological space and \( x, y \) be any two points in \( X \). Then the following statements are equivalent:

\[
\{i\} \quad X \text{ is an } \alpha-\gamma-R_0 \text{ space};
\]

\[
\{ii\} \quad \tau_{\alpha-\gamma} \ker(\{x\}) \subseteq \tau_{\alpha-\gamma} \ker(\{x\});
\]

\[
\{iii\} \quad y \in \tau_{\alpha-\gamma} \ker(\{x\}) \text{ if and only if } x \in \tau_{\alpha-\gamma} \ker(\{y\});
\]

\[
\{iv\} \quad y \in \tau_{\alpha-\gamma} \ker(\{x\}) \text{ if and only if } x \in \tau_{\alpha-\gamma} \ker(\{y\});
\]

\[
\{v\} \quad \text{Let } F \text{ be an } \alpha-\gamma \text{-closed set } F \text{ and a point } x \notin F, \text{ there exists a } U \in \tau_{\alpha-\gamma} \text{ and } F \subseteq U.
\]

\[
\{vi\} \quad \text{Each } \alpha-\gamma \text{-closed set } F \text{ can be expressed as } F = \cap \{U : U \in \tau_{\alpha-\gamma} \text{ and } F \subseteq U\}.
\]
Each $\alpha-\gamma$-open set $U$, $U = \bigcup \{F : X - F \in \tau_{\alpha-\gamma} \text{and} F \subseteq U\}$

Let $F$ be an $\alpha-\gamma$-closed set and $x \notin F$ implies that $\tau_{\alpha-\gamma} \text{-cl}(\{x\}) \cap F = \emptyset$.

**Proof**

$(i) \Rightarrow (ii)$ By the Definition 4.1, $\tau_{\alpha-\gamma} \text{-ker}(\{x\}) = \bigcap \{U : U \in \tau_{\alpha-\gamma} \text{and} \{x\} \subseteq U\}$. Then by $(i)$, $\tau_{\alpha-\gamma} \text{-cl}(\{x\}) \subseteq U$. Hence $\tau_{\alpha-\gamma} \text{-cl}(\{x\}) \subseteq \tau_{\alpha-\gamma} \text{-ker}(\{x\})$.

$(ii) \Rightarrow (iii)$ If $y \in \tau_{\alpha-\gamma} \text{-ker}(\{x\})$, then by Theorem 4.1, $x \in \tau_{\alpha-\gamma} \text{-cl}(\{y\})$. By $(ii)$, $x \in \tau_{\alpha-\gamma} \text{-ker}(\{y\})$. Similarly, if $x \in \tau_{\alpha-\gamma} \text{-ker}(\{y\})$, then $y \in \tau_{\alpha-\gamma} \text{-ker}(\{x\})$.

$(iii) \Rightarrow (iv)$ If $y \in \tau_{\alpha-\gamma} \text{-cl}(\{x\})$, then by Theorem 4.1, $x \in \tau_{\alpha-\gamma} \text{-cl}(\{y\})$. By $(iii)$, $y \in \tau_{\alpha-\gamma} \text{-ker}(\{x\})$. Again by Theorem 4.1, $x \in \tau_{\alpha-\gamma} \text{-cl}(\{y\})$. Similarly, if $x \in \tau_{\alpha-\gamma} \text{-cl}(\{y\})$, then $y \in \tau_{\alpha-\gamma} \text{-cl}(\{x\})$.

$(iv) \Rightarrow (v)$ Let $F$ be an $\alpha-\gamma$-closed set and a point $x \notin F$. Then for any $y \in F$, $\tau_{\alpha-\gamma} \text{-cl}(\{y\}) \subseteq F$ and so $x \notin \tau_{\alpha-\gamma} \text{-cl}(\{y\})$. By $(iv)$, $x \notin \tau_{\alpha-\gamma} \text{-cl}(\{y\})$ implies $y \notin \tau_{\alpha-\gamma} \text{-cl}(\{x\})$, that is there exists a $U_y \in \tau_{\alpha-\gamma}$ such that $y \in U_y$ and $x \notin U_y$. Let $U = \bigcup_{y \in F} \{U_y : y \in \tau_{\alpha-\gamma}, y \in U_y \text{and} x \notin U_y\}$. Then by Theorem 3.4[9], $U$ is an $\alpha-\gamma$-open set such that $x \notin U$ and $F \subseteq U$.

$(v) \Rightarrow (vi)$ Let $F$ be an $\alpha-\gamma$-closed set and $H = \bigcap \{U : U \in \tau_{\alpha-\gamma} \text{and} F \subseteq U\}$. Clearly $F \subseteq H$ and it remains to show that $H \subseteq F$. Let $x \notin F$. Then by $(v)$, there exists an $\alpha-\gamma$-open set $U$ such that $x \notin U$ and $F \subseteq U$ and hence $x \notin H$. Therefore, each $\alpha-\gamma$-closed set $F$ can be expressed as $F = \bigcap \{U : U \in \tau_{\alpha-\gamma} \text{and} F \subseteq U\}$.

$(vi) \Rightarrow (vii)$ The proof follows from the Definitions 3.16 and 3.18[10].

$(vii) \Rightarrow (viii)$ Let $F$ be an $\alpha-\gamma$-closed set and $x \notin F$. Then $X - F = U$ is an $\alpha-\gamma$-open set containing $x$. Then by $(vii)$, $U$ can be written as the union of $\alpha-\gamma$-closed sets and so there is an $\alpha-\gamma$-closed set $H$ such that $x \in F \subseteq U$ and $\tau_{\alpha-\gamma} \text{-cl}(\{x\}) \subseteq U$. Thus $\tau_{\alpha-\gamma} \text{-cl}(\{x\}) \cap F = \emptyset$.

$(viii) \Rightarrow (i)$ Let $U$ be an $\alpha-\gamma$-open set and $x \in U$. Then by $(viii)$, there exists an $\alpha-\gamma$-closed set $F$ such that $x \in F \subseteq U$ and $\tau_{\alpha-\gamma} \text{-cl}(\{x\}) \cap F \neq \emptyset$. Therefore $\tau_{\alpha-\gamma} \text{-cl}(\{x\}) \subseteq F$ and hence $\tau_{\alpha-\gamma} \text{-cl}(\{x\}) \subseteq X$. Thus $X$ is an $\alpha-\gamma-R_0$ space.

**Definition 4.3** A function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is said to be a contra-$\alpha-\gamma$-I-continuous function if $f^{-1}(A)$ is an $\alpha-\gamma$-I-closed set in $X$ for each open set $A$ in $Y$.

**Theorem 4.3** Let $f : (X, \tau, I) \rightarrow (Y, \sigma)$ be a function. Then the following statements are equivalent:

$(i) f$ is a contra-$\alpha-\gamma$-I-continuous function;

$(ii)$ For every closed subset $V$ of $Y$, $f^{-1}(V)$ is an $\alpha-\gamma$-I-open set in $X$;

$(iii)$ For each $x \in X$ and each closed subset $V$ of $Y$ with $f(x) \in V$, there exists an $\alpha-\gamma$-I-open subset $H$ of $X$ with $x \in H$ such that $f(H) \subseteq V$;

$(iv)$ $f^{-1}(\tau_{\alpha-\gamma-I} \text{-cl}(V)) \subseteq \tau_{\alpha-\gamma-I} \text{-ker}(V)$, for every subset $V$ of $X$;

$(v)$ $\tau_{\alpha-\gamma-I} \text{-cl}(f^{-1}(U)) \subseteq f^{-1}(\tau_{\alpha-\gamma-I} \text{-ker}(U))$, for every subset $U$ of $Y$.

**Proof**

$(i) \Rightarrow (ii)$ Proof follows from the Definition 4.3.

$(ii) \Rightarrow (iii)$ Let $V$ be any closed subset of $Y$. If $x \in f^{-1}(V)$, then $f(x) \in V$ and there exists an $\alpha-\gamma$-I-open subset $H_x$ of $X$ with $x \in H_x$ such that $f(H_x) \subseteq V$. Therefore, $f^{-1}(V) = \bigcup \{H_x : x \in f^{-1}(V)\}$. Then $f^{-1}(V)$ is an $\alpha-\gamma$-I-open set.

$(iii) \Rightarrow (iv)$ Let $V$ be any subset of $X$. Suppose that $y \notin \tau_{\alpha-\gamma-I} \text{-ker}(f(V))$. Then by Theorem 4.1, there exists an $\alpha-\gamma$-closed set $U$ of $Y$ containing $V$ such that $f(V) \cap U = \emptyset$. Hence, $V \cap f^{-1}(U) = \emptyset$ and $\tau_{\alpha-\gamma-I} \text{-cl}(V) \cap f^{-1}(U) = \emptyset$. Therefore, $f(\tau_{\alpha-\gamma-I} \text{-cl}(V) \cap U = \emptyset$.

281
and $y \notin f(\tau_{\alpha-\gamma-I}(\text{cl}(V)))$. This implies that $f(\tau_{\alpha-\gamma-I}(\text{cl}(V))) \subseteq \tau_{\alpha-\gamma-I}\text{-ker}(f(V))$.

\{iv\} $\Rightarrow$ \{v\} Let $U$ be any subset of $Y$. By hypothesis and Theorem 4.1, $f(\tau_{\alpha-\gamma-I}(\text{cl}(f^{-1}(U)))) \subseteq \tau_{\alpha-\gamma-I}\text{-ker}(f(U))$ and $\tau_{\alpha-\gamma-I}(\text{cl}(f^{-1}(U))) \subseteq f^{-1}(\tau_{\alpha-\gamma-I}\text{-ker}(U))$.

\{v\} $\Rightarrow$ \{i\} Let $G$ be any open set of $Y$. Then by Theorem 4.2, $\tau_{\alpha-\gamma-I}(\text{cl}(f^{-1}(G))) \subseteq f^{-1}(\tau_{\alpha-\gamma-I}\text{-ker}(G)) = f^{-1}(G)$ and $\tau_{\alpha-\gamma-I}(\text{cl}(f^{-1}(G))) = f^{-1}(G)$. This shows that $f^{-1}(G)$ is an $\alpha\gamma$-$I$-closed set in $(X, \tau, I)$. Hence $f$ is a contra-$\alpha\gamma$-$I$-continuous function.

**Theorem 4.4** Let $(X, \tau, I)$ be an ideal topological space. Then

\{i\} Every contra-$\alpha\gamma$-$I$-continuous function is a contra-$\alpha\gamma$-$I$-continuous function.

\{ii\} Every contra-continuous function is an $\alpha\gamma$-$I$-continuous function.

**Proof** The proof follows from the Definition 4.3.

**Remark 4.1** The converse of the above Theorem 4.3 need not be true as shown by the following examples.

**Example 4.1** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$, \{a, b, c\}, \{a, b, d\}$, $\sigma = \{\phi, Y, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$, \{a, b, c\}, \{a, b, d\}$ and $I = \{\phi, \{\} \}$. Define $\gamma : \tau \rightarrow P(X)$ as follows: for every $A \in \tau$,

$$A^\gamma = \begin{cases} \text{int} (\text{cl}(A)) & \text{if } A \neq \{\} \\ \text{cl}(A) & \text{if } A = \{\} \end{cases}$$

Then $\tau_{\alpha-\gamma} = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$ and $\tau_{\alpha-\gamma-I} = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, d\}\}$. Define a function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ as $f(a) = d$, $f(b) = c$, $f(c) = b$, $f(d) = a$. Then for every $V \in \sigma$, $f^{-1}(V)$ is an $\alpha\gamma$-$I$-closed set in $(X, \tau, I)$. Hence $f$ is a contra-$\alpha\gamma$-$I$-continuous function. But $f^{-1}(\{d\}) = \{a\} \notin (X, \tau, I)$. Hence $f$ is not a contra-$\alpha\gamma$-$I$-continuous function.

**Example 4.2** Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, c\}\}$ and $I = \{\phi, \{\} \}$. Define an operation $\gamma : \tau \rightarrow P(X)$ as follows: for every $A \in \tau$,

$$A^\gamma = \begin{cases} A \cup \{c\} & \text{if } A \neq \{\} \\ A & \text{if } A = \{\} \end{cases}$$

Then $\tau_{\alpha-\gamma} = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and $\tau_{\alpha-\gamma-I} = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$. Let $Y = \{a, b, c\}$, $\sigma = \{\phi, Y, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Define $f : (X, \tau, I) \rightarrow (Y, \sigma)$ as $f(a) = c$, $f(b) = b$, $f(c) = a$. Then for every $V \in \sigma$, $f^{-1}(V)$ is an $\alpha\gamma$-$I$-open set in $(X, \tau, I)$. Hence $f$ is an $\alpha\gamma$-$I$-continuous function. But $f^{-1}(\{a, c\}) = \{c, a\}$, $f^{-1}(\{b, c\}) = \{b, a\}$ are not closed sets in $(X, \tau)$. Hence $f$ is not a contra-continuous function.

**Remark 4.2** The concept of $\alpha\gamma$-$I$-continuity and contra-$\alpha\gamma$-$I$-continuity are independent of each other.

The following examples show the result.

**Example 4.3** Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $I = \{\phi, \{\} \}$. 

6
Define an operation \( \gamma : \tau \rightarrow P(X) \) as follows: for every \( B \in \tau \), \( \gamma(B) = \text{cl}(B) \). Then \( \tau_{\alpha - \gamma} = \{ \phi, X, \{b\}, \{a, c\} \} \) and \( \tau_{\alpha - \gamma - I} = \{ \phi, X, \{b\}, \{a, c\} \} \).

Let \( Y = \{a, b, c\} \), \( \sigma = \{ \phi, Y, \{a\}, \{c\} \} \), \( \{a, b\}, \{a, c\} \}. Define \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) as \( f(a) = a ; f(b) = c ; f(c) = b \). Then for every \( V \in \sigma \), \( f^{-1}(V) \) is an \( \alpha - \gamma - I \)-open set in \( (X, \tau, I) \). Hence \( f \) is an \( \alpha - \gamma - I \)-continuous function. But \( f \) is not a contra-\( \alpha - \gamma - I \)-continuous function.

**Example 4.3** Let \( X = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \) and \( I = \{\phi, \{b\}\} \).

Define an operation \( \gamma : \tau \rightarrow P(X) \) as follows: for every \( A \in \tau \),

\[
A^\gamma = \begin{cases} 
A \cup \{c\} & \text{if } A \neq \{a\} \\
A & \text{if } A = \{a\}
\end{cases}
\]

Then \( \tau_{\alpha - \gamma} = \{\phi, X, \{a\}, \{c\}, \{a, c\}\} \) and \( \tau_{\alpha - \gamma - I} = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \).

Let \( Y = \{a, b, c\} \), \( \sigma = \{\phi, Y, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \) Define \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) as \( f(a) = c \); \( f(b) = a \); \( f(c) = b \). Then for every \( V \in \sigma \), \( f^{-1}(V) \) is an \( \alpha - \gamma - I \)-closed set in \( (X, \tau, I) \). Hence \( f \) is a contra-\( \alpha - \gamma - I \)-continuous function. But \( f \) is not an \( \alpha - \gamma - I \)-continuous function.

**Definition 4.4** A topological space \( (X, \tau, I) \) is called an \( \alpha - \gamma - I \)-\( T_2 \) space if for each disjoint points \( x \) and \( y \) in \( X \), there exists -open sets \( G \) and \( H \) in \( X \) such that \( x \in G \), \( y \in H \) and \( G \cap H = \phi \).

**Definition 4.5** A topological space \( (X, \tau, I) \) is said to be an \( \alpha - \gamma - I \)-connected space if there does not exist a pair \( A, B \) of non empty disjoint \( \alpha - \gamma - I \)-open subsets of \( X \) such that \( X = A \cup B \), otherwise \( X \) is called an \( \alpha - \gamma - I \)-disconnected space. The pair \( (A, B) \) is called an \( \alpha - \gamma - I \)-disconnection of \( X \).

**Definition 4.6** A collection \( \Omega \) of subsets of \( X \) is said to be an \( \alpha - \gamma - I \)-open cover of \( X \) if the union of elements of \( \Omega \) is equal to \( X \) and its elements are \( \alpha - \gamma - I \)-open sets.

**Definition 4.7** A topological space \( (X, \tau, I) \) is said to be an \( \alpha - \gamma - I \)-compact space if every \( \alpha - \gamma - I \)-open cover of \( X \) has a finite subcollection that also covers \( X \).

**Definition 4.8** A subset \( K \) of \( X \) is said to be an \( \alpha - \gamma - I \)-compact set if for every \( \alpha - \gamma - I \)-open cover of \( X \) there exists a finite subfamily \( F_1, F_2, \ldots, F_n \) of \( \Omega \) such that \( K \subseteq \bigcup_{i=1}^{n} F_i \).

**Definition 4.9** A topological space \( (X, \tau, I) \) is said to be an \( \alpha - \gamma - I \)-regular space, if for any \( \alpha - \gamma - I \)-closed set \( A \) and \( x \notin A \), there exist \( \alpha - \gamma - I \)-open sets \( U, V \) such that \( x \in U \), \( A \subseteq V \) and \( U \cap V = \phi \).

**Definition 4.10** A topological space \( (X, \tau, I) \) is said to be an \( \alpha - \gamma - I \)-normal space, if for any distinct \( \alpha - \gamma - I \)-closed sets \( A \) and \( B \) of \( X \), there exists \( \alpha - \gamma - I \)-open sets \( U, V \) such that \( A \subseteq U \) and \( B \subseteq V \) and \( U \cap V = \phi \).

**Theorem 4.5** If a function \( f : X \rightarrow Y \) is a contra-\( \alpha - \gamma - I \)-continuous function and \( Y \) is
an $\alpha$-$\gamma$I-regular space, then $f$ is an $\alpha$-$\gamma$I-continuous function.

**Proof** Let $x \in X$ and $G$ be an open subset of $Y$ with $f(x) \in G$. Since $Y$ is an $\alpha$-$\gamma$I-regular space, there exists an $\alpha$-$\gamma$I-open set $H$ in $Y$ such that $f(x) \in H \subseteq \tau_{\alpha\gamma\text{-I}}(H) \subseteq G$. But $f$ is a contra $\alpha$-$\gamma$I-continuous function, by Theorem 3.1, there exists an $\alpha$-$\gamma$I-open set $V$ in $X$ with $x \in V$ such that $f(V) \subseteq \tau_{\alpha\gamma\text{-I}}(H)$. Then $f(V) \subseteq \tau_{\alpha\gamma\text{-I}}(H) \subseteq G$. Hence, $f$ is an $\alpha$-$\gamma$I-continuous function.

**Remark 4.3** If a function $f : X \to Y$ is an $\alpha$-$\gamma$I-continuous function and $Y$ is an $\alpha$-$\gamma$I-regular space, then $f$ need not be a contra $\alpha$-$\gamma$I-continuous function as shown in the example (i) of Remark 4.2.

**Remark 4.3** The following example shows that the composition of any two contra-$\alpha$-$\gamma$I-continuous functions need not be a contra-$\alpha$-$\gamma$I-continuous function.

(i) Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ and $I = \{\phi, \{b\}\}$. Define $\gamma : \tau \to P(X)$ as follows: for every $A \in \tau$,

$$A^\gamma = \begin{cases} A \cup \{c\} & \text{if } A \neq \{a\} \\ A & \text{if } A = \{a\} \end{cases}$$

Then $\tau_{\alpha\gamma} = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ and $\tau_{\alpha\gamma\text{-I}} = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$.

Let $Y = \{a, b, c\}$, $W = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $K = \{\phi, \{a\}\}$. Define $f$ as $f(a) = c; f(b) = a; f(c) = b$. Then for every $V \in \sigma$, $f^{-1}(V)$ is an $\alpha$-$\gamma$I-open set in $(X, \tau, I)$. Hence $f$ is a contra-$\alpha$-$\gamma$I-continuous function. But $f$ is not an $\alpha$-$\gamma$I-continuous function.

(ii) Let $Y = \{a, b, c\}$, $\sigma = \{\phi, Y, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ and $K = \{\phi, \{b\}\}$. Define an operation $\gamma : \sigma \to P(Y)$ as follows: for every $A \in \sigma$,

$$A^\gamma = \begin{cases} A \cup \{c\} & \text{if } A \neq \{a\} \\ A & \text{if } A = \{a\} \end{cases}$$

Then $\sigma_{\alpha\gamma\text{-K}} = \{\phi, y, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$.

Let $Z = \{a, b, c\}$, $\zeta = \{\phi, Z, \{a\}, \{a, b\}, \{a, c\}\}$ and $W = \{\phi, \{b\}\}$. Define $g$ as $g(a) = b; g(b) = c; g(c) = a$. Then for every $V \in \zeta$, $g^{-1}(V)$ is an $\alpha$-$\gamma$I-open set in $(Y, \sigma, K)$.

**Theorem 4.6** Let $f : (X, \tau, I) \to (Y, \sigma, K)$ and $g : (Y, \sigma, K) \to (Z, \zeta)$. Then:
\{i\} \ g \circ f \ is \ a \ contra-\alpha-\gamma-I\text{-continuous} \ function, \ if \ g \ is \ a \ continuous \ function \ and \ f \ is \ a \ contra-\alpha-\gamma-I\text{-continuous} \ function; \\
\{ii\} \ g \circ f \ is \ a \ contra-\alpha-\gamma-I\text{-continuous} \ function, \ if \ g \ is \ a \ contra-continuous \ function \ and \ f \ is \ a \ contra-\alpha-\gamma-I\text{-continuous} \ function;

**Definition 4.11** A function \( f : (X, \tau, I) \to (Y, \sigma) \) is said to satisfy the \( \alpha-\gamma-I\text{-interiority} \) condition, if \( \tau_{\alpha-\gamma-I\text{-int}}(f^{-1}(cl(G))) \subseteq f^{-1}(G) \), for each open set \( G \) of \( (Y, \sigma) \).

**Theorem 4.7** If \( f : (X, \tau, I) \to (Y, \sigma) \) is a contra-\( \alpha-\gamma-I\text{-continuous} \) function and satisfies \( \alpha-\gamma-I\text{-interiority} \) condition, then \( f \) is an \( \alpha-\gamma-I\text{-continuous} \) function.

**Proof** Let \( G \) be any open set of \( Y \). Since \( f \) is a contra-\( \alpha-\gamma-I\text{-continuous} \) function and \( cl(G) \) is closed, by Theorem 4.2, \( f^{-1}(cl(G)) \) is an \( \alpha-\gamma-I\text{-open} \) set in \( X \). By hypothesis of \( f \), \( f^{-1}(G) \subseteq f^{-1}(cl(G)) \)). Therefore, \( f^{-1}(G) = \tau_{\alpha-\gamma-I\text{-int}}(f^{-1}(cl(G))) \). This implies that \( f^{-1}(G) \in \tau_{\alpha-\gamma-I} \). This shows that \( f \) is an \( \alpha-\gamma-I\text{-continuous} \) function.

**Theorem 4.8** If \( (X, \tau, I) \) is an ideal topological space and for each pair of distinct points \( x_1 \) and \( x_2 \) in \( X \), there exists a function \( f : (X, \tau, I) \to (Y, \sigma) \), where \( (Y, \sigma) \) is a Urysohn space such that \( f(x_1) \neq f(x_2) \) and \( f \) is contra-\( \alpha-\gamma-I\text{-continuous} \) at \( x_1 \) and \( x_2 \), then \( (X, \tau, I) \) is an \( \alpha-\gamma-I-T_2 \) space.

**Proof** Let \( x_1 \) and \( x_2 \) be any distinct points in \( X \). Then by hypothesis there is a Urysohn space \( (Y, \sigma) \) and a function \( f : (X, \tau, I) \to (Y, \sigma) \) which satisfies the condition of this theorem. Let \( y_i = f(x_i) \), for \( i = 1, 2 \). Then \( y_1 \neq y_2 \). Since \( (Y, \sigma) \) is a Urysohn space, there exist open neighbourhoods \( H_{y_1} \) and \( H_{y_2} \) of \( y_1 \) and \( y_2 \) respectively in \( y \) such that \( cl(H_{y_1}) \cap cl(H_{y_2}) = \phi \). Since \( f \) is a contra-\( \alpha-\gamma-I\text{-continuous} \) function at \( x_i \), there exist \( \alpha-\gamma-I\text{-open} \) neighbourhoods \( V_{x_i} \) of \( x_i \) in \( X \) such that \( f(V_{x_i}) \subseteq cl(H_{y_i}) \) for \( i = 1, 2 \). Hence \( V_{x_1} \cap V_{x_2} = \phi \). Then \( (X, \tau, I) \) is an \( \alpha-\gamma-I-T_2 \) space.

**Theorem 4.9** Let \( f : (X, \tau, I) \to (Y, \sigma) \) be a contra-\( \alpha-\gamma-I\text{-continuous} \) injective function. If \( Y \) is a Urysohn space, then \( X \) is an \( \alpha-\gamma-I-T_2 \) space.

**Proof** Let \( x \) and \( y \) be a pair of disjoint points in \( X \). Then \( f(x) \neq f(y) \). Since \( Y \) is a Urysohn space, there exist open sets \( G \) and \( H \) of \( Y \) such that \( f(x) \in G \), \( f(y) \in H \) and \( cl(G) \cap cl(H) = \phi \). Since \( f \) is a contra-\( \alpha-\gamma-I\text{-continuous} \) function at \( x \) and \( y \), there exist \( \alpha-\gamma-I\text{-open} \) sets \( U \) and \( V \) in \( X \) such that \( x \in U \), \( y \in V \) and \( f(U) \subseteq cl(G) \), \( f(V) \subseteq cl(H) \). Then, \( f(U) \cap f(V) = \phi \). So \( U \cap V = \phi \). Hence \( X \) is an \( \alpha-\gamma-I-T_2 \) space.

**Theorem 4.10** If \( f : (X, \tau, I) \to (Y, \sigma) \) is a contra-\( \alpha-\gamma-I\text{-continuous} \) injective function and \( (Y, \sigma) \) is an ultra Hausdorff space, then \( (X, \tau) \) is an \( \alpha-\gamma-I-T_2 \) space.

**Proof** Let \( x_1 \) and \( x_2 \) be distinct points in \( X \). Since \( f \) is an injective function and \( (Y, \sigma) \) is an ultra Hausdorff space, \( f(x_1) \neq f(x_2) \), there exist clopen sets \( G_1, G_2 \) such that \( f(x_1) \in G_1 \), \( f(x_2) \in G_2 \) and \( G_1 \cap G_2 = \phi \). Then \( x_i \in f^{-1}(G_i) \in \tau_{\alpha-\gamma-I} \), for \( i = 1, 2 \) and \( f^{-1}(G_1) \cap f^{-1}(G_2) = \phi \).
φ. Thus \((X, \tau, I)\) is an \(\alpha\gamma\I\T_2\)space.

**Theorem 4.11** If \(f : (X, \tau, I) \to (Y, \sigma)\) is a contra-\(\alpha\gamma\I\)continuous, closed injective function and \(Y\) is an ultra normal space, then \(X\) is an \(\alpha\gamma\I\)normal space.

**Proof** Let \(V_1\) and \(V_2\) be disjoint closed subsets of \(X\). Since \(f\) is a closed and injective function, \(f(V_1)\) and \(f(V_2)\) are disjoint closed subsets of \(Y\). But \(Y\) is an ultra normal space, hence \(f(V_1)\) and \(f(V_2)\) are separated by disjoint clopen sets \(G_1\) and \(G_2\) respectively. Since \(f\) is a contra-\(\alpha\gamma\I\)continuous function, \(f^{-1}(G_1)\) and \(f^{-1}(G_2)\) are \(\alpha\gamma\I\)-open sets, with \(V_1 \subseteq f^{-1}(G_1)\), \(V_2 \subseteq f^{-1}(G_2)\) and \(f^{-1}(G_1) \cap f^{-1}(G_2) = \varnothing\). Hence, \(X\) is an \(\alpha\gamma\I\)-normal space.

**Theorem 4.12** If \(f : (X, \tau, I) \to (Y, \sigma)\) is a contra-\(\alpha\gamma\I\)-continuous function from a \(\alpha\gamma\I\)-connected space \(X\) onto any space \(Y\), then \(Y\) is not a discrete space.

**Proof** Suppose that \(Y\) is a discrete space. Let \(A\) be a proper non-empty clopen set in \(Y\). Then \(f^{-1}(A)\) is a proper non-empty \(\alpha\gamma\I\)-clopen subset of \(X\) which contradicts the fact that \(X\) is an \(\alpha\gamma\I\)-connected space.

**Theorem 4.13** A contra-\(\alpha\gamma\I\)-continuous image of an \(\alpha\gamma\I\)-connected space is a connected space.

**Proof** Let \(f : (X, \tau, I) \to (Y, \sigma)\) be a contra-\(\alpha\gamma\I\)-continuous function from an \(\alpha\gamma\I\)-connected space \(X\) onto any space \(Y\). Assume that \(Y\) is a disconnected space. Then \(Y = A \cup B\), where \(A\) and \(B\) are non-empty clopen sets in \(Y\) with \(A \cup B = \varnothing\). Since \(f\) is a contra-\(\alpha\gamma\I\)-continuous function, \(f^{-1}(A)\) and \(f^{-1}(B)\) are non-empty, \(\alpha\gamma\I\)-open sets in \(X\) with \(f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X\) and \(f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\varnothing) = \varnothing\). This means that \(X\) is not an \(\alpha\gamma\I\)-connected space which is a contradiction. Then \(Y\) is a connected space.

**Theorem 4.14** Let \(X\) be an \(\alpha\gamma\I\)-connected space and \(Y\) is a \(T_1\)-space. If \(f : (X, \tau, I) \to (Y, \sigma)\) is a contra-\(\alpha\gamma\I\)-continuous function, then \(f\) is a constant.

**Proof** Let \(\Delta = \{f^{-1}(y) : y \in Y\}\). Since \(Y\) is a \(T_1\)-space implies that, \(\Delta\) is an \(\alpha\gamma\I\)-disjoint partition of \(X\). Hence if \(|\Delta| \geq 2\), then \(X\) is the union of two non-empty open sets. Since \((X, \tau, I)\) is an connected space, \(|\Delta| = 1\). Therefore, \(f\) is constant.

**Remark 4.5** Every contra \((\gamma, \beta)-(I,K)\)-continuous function is a contra-\(\alpha\gamma\I\)-continuous function, but the converse is not true as shown by the following example.

Let \(X = \{a, b, c, \}, \tau = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}\) and \(I = \{\varnothing, \{b\}\}\). Define \(\gamma : \tau \to P(X)\) as follows: for every \(A \in \tau\),

\[
A^\gamma = \begin{cases} 
A \cup \{c\} & \text{if } A \neq \{a\} \\
A & \text{if } A = \{a\}
\end{cases}
\]

Then \(\tau_{\alpha\gamma\I} = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}\).
Let $Y = \{a, b, c\}$, $\sigma = \{\phi, Y, \{a\}, \{a,b\}, \{a,c\}\}$ and $K = \{\phi, \{b\}\}$. Define an operation $\beta : \sigma \to P(Y)$ as follows: for every $A \in \sigma$,

$$A^\gamma = \begin{cases} A \cup \{c\} & \text{if } A \neq \{a\} \\ A & \text{if } A = \{a\} \end{cases}$$

Then $\sigma_{\alpha-\beta-K} = \{\phi, Y, \{a\}, \{a\}, \{a,b\}, \{a,c\}\}$.

Define $f$ as $f(a) = b$; $f(b) = c$; $f(c) = a$. Then for every $V \in \sigma$, $f^{-1}(V)$ is an $\alpha-\gamma$-I-open set in $(X, \tau, I)$. Hence $f$ is a contra-$\alpha-\gamma$-I-continuous function. But $f^{-1}(c) = a$, $f^{-1}(a, b) = c, a$ are not $\alpha-\gamma$-I-closed sets in $X$. Hence $f$ is not a contra $\alpha-(\gamma, \beta)$-(I,K)-continuous function.

## 5 $\alpha-\gamma$-I-Hausdorff space

**Definition 5.1** Let $(X, \tau, I)$ be an ideal topological space and $\gamma$ be an operation on $\tau$. Then $(X, \tau, I)$ is said to be an $\alpha-\gamma$-I-Hausdorff space if for each two distinct points $x, y$ there exists $\alpha-\gamma$-I-open sets $U$ and $V$ containing $x$ and $y$ respectively such that $U \cap V = \phi$. Then the points $x$ and $y$ are said to be $\alpha-\gamma$-I-separated.

**Theorem 5.1** Every $\alpha-\gamma$-I-Hausdorff space is an $\alpha-\gamma$-Hausdorff space (resp. $\gamma$-Hausdorff space).

**Proof** The proof follows from the Definition 3.1, Theorem 3.3.[9] and Definition 3.1, Theorem 3.4.[10].

The converse of the above theorem need not be true follows from the example 5.1.

**Example 5.1** Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ and define an operation $\gamma : \tau \to P(X)$ as follows: for every $A \in \tau$

$$A^\gamma = \begin{cases} A \cup \{c\} & \text{if } A \neq \{a\}, \{b\} \text{ or } \{c\} \\ A & \text{otherwise} \end{cases}$$

Then $\tau_{\alpha-\gamma} = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}\}$. Hence $(X, \tau)$ is an $\alpha-\gamma$-Hausdorff space.

(i) If $I = \{\phi, \{a\}\}$ then $\tau_{\alpha-\gamma} = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a,c\}\}$. Hence $(X, \tau, I)$ is an $\alpha-\gamma$-I-Hausdorff space and $(X, \tau)$ is an $\alpha-\gamma$-Hausdorff space (resp. $\gamma$-Hausdorff space).

(ii) If $I = \{\phi\}$ or $I = P(X)$, then $(X, \tau, I)$ is an $\alpha-\gamma$-I-Hausdorff space and $(X, \tau)$ is an $\alpha-\gamma$-Hausdorff space (resp. $\gamma$-Hausdorff space).

**Theorem 5.2** Let $(X, \tau, I)$ be an ideal topological space. Then

(i) Let $I = \{\phi\}$. Then $(X, \tau, I)$ is an $\alpha-\gamma$-I-Hausdorff space if and only if it is an $\alpha-\gamma$-Hausdorff space.

(ii) Let $I = P(X)$. Then $(X, \tau, I)$ is a $\gamma$-Hausdorff space if and only if it is an $\alpha-\gamma$-I-Hausdorff space.

11
Proof \{i\} Let I = \{\phi\}. Then \(A^* = \tau_\gamma \text{cl}(A)\) and \(\tau_\gamma \text{cl}^*(A) = \tau_\gamma \text{cl}(A)\) for every subset A of X. Therefore, \(\tau_{\alpha-\gamma} \subseteq \tau_{\alpha-\gamma}^{*}\) and hence \((X,\tau,I)\) is \(\alpha-\gamma\)-Hausdorff if and only if it is an \(\alpha-\gamma\)-Hausdorff space.

\{ii\} Let I = P(X). Then \(A^* = \phi\) and \(\tau_\gamma \text{cl}^*(A) = A\) for every subset A of X. Let A \(\subseteq \tau_{\alpha-\gamma} I\), then A \(\subseteq \tau_\gamma \text{int}(\tau_\gamma \text{cl}^*(\tau_\gamma \text{int}(A))) = \tau_\gamma \text{int}(A)\) and hence A is open in \((X,\tau)\). Therefore, \((X,\tau,I)\) is a \(\gamma\)-Hausdorff space if and only if it is an \(\alpha-\gamma\)-I-Hausdorff space.

Definition 5.2 An ideal topological space \((X,\tau,I)\) is called an \(\alpha-\gamma\)-complete space if \(\tau^* = \tau_{\alpha-\gamma}^{*}\), that is a subset A of X is a \(\tau^*\)-open set if and only if it is an \(\alpha-\gamma\)-I-open set.

Theorem 5.3 Let \((X,\tau,I_n)\) be an ideal topological space, where \(I_n\) is the ideal of the nowhere dense sets of \((X,\tau)\). Then

\{i\} \((X,\tau,I_n)\) is an \(\alpha-\gamma\)-I-Hausdorff space if and only if it is an \(\alpha-\gamma\)-I-complete space.

\{ii\} If \((X,\tau,I_n)\) is an \(\alpha-\gamma\)-Hausdorff space and \(\alpha-\gamma\)-I-complete, then it is a Hausdorff space.

Proof \{i\} Since \(I_n\) is the ideal of nowhere dense sets of \((X,\tau)\), then \(A^* = \tau_\gamma \text{cl}(\tau_\gamma \text{int}(\tau_\gamma \text{cl}(A)))\) and hence by example 3.10 of [9] \(\tau_\gamma \text{cl}^*(A) = A \cup \tau_\gamma \text{cl}(\tau_\gamma \text{int}(\tau_\gamma \text{cl}(A))) = \tau_\gamma \text{cl}(A)\).

For every subset A of X, \(\tau_\gamma \text{cl}^*(\tau_\gamma \text{int}(A)) = \tau_\gamma \text{int}(A) \cup \tau_\gamma \text{cl}(\tau_\gamma \text{int}(\tau_\gamma \text{cl}(\tau_\gamma \text{int}(A)))) = \tau_\gamma \text{int}(A) \cup \tau_\gamma \text{int}(\tau_\gamma \text{cl}(\tau_\gamma \text{int}(A)))\).

Therefore, A \(\subseteq \tau_{\alpha-\gamma} I\) if and only if A \(\subseteq \tau_{\alpha-\gamma}^{*}\). It follows that \((X,\tau,I_n)\) is an \(\alpha-\gamma\)-I-Hausdorff space if and only if it is an \(\alpha-\gamma\)-Hausdorff space.

\{ii\} Let \((X,\tau,I_n)\) be an \(\alpha-\gamma\)-Hausdorff and \(\alpha-\gamma\)-I-complete space. Then A \(\subseteq \tau_{\alpha-\gamma} I\) if and only if A \(\subseteq \tau_{\alpha-\gamma}^{*}\) if and only if A is an \(\alpha-\gamma\)-I-open set. Hence from \{\bar{i}\}, \(\tau_{\alpha-\gamma} = \tau_{\alpha-\gamma}^{*}\) and then \((X,\tau,I_n)\) is a Hausdorff space.

Lemma 5.1 Let I and K be two ideals on a topological space \((X,\tau)\) and I \(\subseteq\) K. Then the following properties hold:

\{i\} \(\tau_{\gamma-K} \text{cl}^*(A) \subseteq \tau_{\gamma-I} \text{cl}^*(A)\) for each subset A of X,

\{ii\} \(\tau_{\alpha-\gamma-K} \subseteq \tau_{\alpha-\gamma-I}\).

Proof \{i\} If I \(\subseteq\) K, then \(A^*(K) \subseteq A^*(I)\) and \(\tau_{\gamma-K} \text{cl}^*(A) = A \cup A^*(K) \subseteq A \cup A^*(I) = \tau_{\gamma-I} \text{cl}^*(A)\) for each subset A of X.

\{ii\} Let A \(\subseteq \tau_{\alpha-\gamma-K}\). Then A \(\subseteq \tau_\gamma \text{int}(\tau_{\gamma-K} \text{cl}^*(\tau_\gamma \text{int}(A))) = \tau_\gamma \text{int}(\tau_{\gamma-I} \text{cl}^*(\tau_\gamma \text{int}(A)))\) and hence A \(\subseteq \tau_{\alpha-\gamma-I}\).

Theorem 5.3 Let I and K be two ideals on a topological space \((X,\tau)\) and I \(\subseteq\) K. If \((X,\tau,K)\) is an \(\alpha-\gamma\)-K-Hausdorff space, then \((X,\tau,I)\) is an \(\alpha-\gamma\)-I-Hausdorff space.

Proof Proof follows from the Lemma 3.1.
Lemma 5.2 Let \((X, \tau, I)\) be an ideal topological space. If \(U \in \tau\) and \(V \in \tau_{\alpha-\gamma-I}\), then 
\(U \cap V \in \tau_{\alpha-\gamma-I(U, \tau|U, I|U)}\).

Proof The proof follows from the Definition 3.2 [13].

Theorem 5.4 Let \((X, \tau, I)\) be an \(\alpha-\gamma-I\)-Hausdorff space and \(A \subseteq X\) be a \(\gamma\)-open set, then 
\((A, \tau_A, I_A)\) is an \(\alpha-\gamma-I\)-Hausdorff space.

Proof The proof follows from the Lemma 5.2.

Theorem 5.5 Let \(f : (X, \tau, I) \to (Y, \sigma, K)\) be an \(\alpha-(\gamma, \beta)-(I,K)\)-continuous injection from a space \(X\) into \(Y\). If \((Y, \sigma, K)\) is an \(\alpha-\beta-K\)-Hausdorff space, then \(X\) is an \(\alpha-\gamma-I\)-Hausdorff space.

Proof Let \(x, y \in X\) and \(x \neq y\). Then \(f(X) \neq f(y)\) thus \(f(x)\) and \(f(y)\) are \(\alpha-\beta-K\)-separated in \(Y\) by \(\alpha-\beta-K\)-open sets \(U\) and \(V\) respectively. Since \(f\) is an \(\alpha-(\gamma, \beta)-(I,K)\)-continuous function, \(f^{-1}(U)\) and \(f^{-1}(V)\) are disjoint \(\alpha-\gamma-I\)-open sets containing \(x\) and \(y\) respectively. This shows that \(X\) is an \(\alpha-\gamma-I\)-Hausdorff space.

Theorem 5.6 Let \((X, \tau, I)\) be an ideal topological space with the following property: If \(x \neq y\) then there exist a Hausdorff space \((Y, \sigma)\) and an \(\alpha-\gamma-I\)-continuous function \(f : (X, \tau, I) \to (Y, \sigma)\) such that \(f(x) \neq f(y)\). Then \((X, \tau, I)\) is an \(\alpha-\gamma-I\)-Hausdorff space.

Proof Proof follows from the Definition 3.1 [10].

References


[21] N. Kalaivani, A-I-EL-Maghrabi and D. Saravanakumar, Certain Separation Axioms and $R_1$ Spaces with $\alpha$-$\gamma$-Open sets in topological spaces, (Submitted)


