

Quotient Labeling of Shadow Graph and Middle Graph of Path and Cycles and two types of Caterpillar graph

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Abstract

Let $G(V, E)$ be a finite, simple, non-trivial and undirected graph of order n and size m . For an one to one assignment $f : V(G) \rightarrow \{1, 2, \dots, n\}$, a quotient labeling $f^* : E(G) \rightarrow \{1, 2, \dots, n\}$ defined by $f^*(uv)$ is equal to the lower bound of $f(u)$ divided by $f(v)$, where $f(u) > f(v)$, then the edge labels need not be distinct. The q -labeling number $q_l(f^*)$ is the maximum value of $f^*(E(G))$, and the Quotient Labeling Number $Q_L(G)$ is the minimum among $q_l(f^*)$. The bounds for the Quotient labeling number for middle graph of path and cycles, shadow graph of path and cycles and two types of caterpillar Graph are found in this paper.

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Key Words and Phrases: Quotient labeling number; middle graph; shadow graph; caterpillar graph.

1 Introduction

In earlier, Alex Rosa introduced the graph labeling problems in the year 1967. Every year a survey comes with the updating of different types graph labeling problems by J. A. Gallian (see[1]). Quotient labeling of graphs was first introduced by P. Sumathi and A. Rathi. They found the quotient labeling number of various graphs and are found in (see[2-6]). The quotient labeling number of middle graph of path and cycles, shadow graph of path and cycles and two types of caterpillar graph are found here. For Graph notations and terminology we follow Harary (see[7]).

2 preliminaries

Definition 1. The middle graph of a graph G is a graph $M(G)$ with $V(G) = \{v_i \cup e_j : 1 \leq i \leq n, 1 \leq j \leq n-1\}$ and if any pair of vertices are adjacent in $M(G)$ if and only if either they are adjacent or incident in G .

Definition 2. The shadow graph of a graph G is a graph $D_2(G)$ is obtained from two copies G_1 and G_2 of G with $V(G) = \{v_i \cup u_i : 1 \leq i \leq n\}$ and join the vertex v in G_1 with the neighbor vertices of u in G_2 .

3 Main Result

Definition 3. Let $G(V, E)$ be a finite, simple, non-trivial and undirected graph of order n and size m . For an one to one assignment $f : V(G) \rightarrow \{1, 2, \dots, n\}$, a quotient labeling $f^* : E(G) \rightarrow \{1, 2, \dots, n\}$ defined by $f^*(uv)$ is equal to the lower bound of $f(u)$ divided by $f(v)$, where $f(u) > f(v)$, then the edge labels need not be distinct. The q -labeling number $q_l(f^*)$ is the maximum value of $f^*(E(G))$, and the Quotient Labeling Number $Q_L(G)$ is the minimum among $q_l(f^*)$.

Theorem 4. The quotient labeling number of $M(P_n)$ is 2.

Proof. Let $G = M(P_n)$. Let $v_1, v_2, \dots, v_n$ be the n vertices and $e_1, e_2, \dots, e_{n-1}$ be the $n-1$ edges of the path P_n . let $u_1, u_2, \dots, u_{n-1}$ be

the $n - 1$ vertices corresponding to the $n - 1$ edges $e_1, e_2, \dots, e_{n-1}$ of P_n . Now G has $2n - 1$ vertices with $\deg(v_i) = 1$ for $i = 1$ and n , $\deg(v_i) = 2$ for $2 \leq i \leq n - 1$, $\deg(u_1) = \deg(u_{n-1}) = 3$ and $\deg(u_i) = 4$ for $2 \leq i \leq n - 2$. Define the function $f : V(G) \rightarrow \{1, 2, \dots, 2n - 1\}$ as follows: $f(v_i) = 2i - 1$ for $1 \leq i \leq n$, $f(u_i) = 2i$ for $1 \leq i \leq n - 1$. For the above vertex labeling we get $f^*(E(G)) = \{1, 2\}$. Then maximum value of $f^*(E(G))$ is equal to 2. But in G , $\delta(G) = 1$ and $\Delta(G) = 4$. Therefore $q_l(f^*)$ can take the value 2 or 3 or 4 or 5. Here $q_l(f^*) = 2$ and it is minimum. Hence $Q_L(M(P_n)) = 2$. $\square$

Theorem 5. *The quotient labeling number of $M(C_n)$ is 3.*

Proof. Let $G = M(C_n)$. Let $v_1, v_2, \dots, v_n$ be the n vertices and let $e_1, e_2, \dots, e_n$ be the n edges of the cycle C_n . Let $u_1, u_2, \dots, u_n$ be the n vertices corresponding to the n edges $e_1, e_2, \dots, e_n$ of C_n to obtain G . Now G has $2n$ vertices with $\deg(v_i) = 2$ for $1 \leq i \leq n$ and $\deg(u_i) = 4$ for $1 \leq i \leq n$. We prove this theorem on two different cases on n . Case (i) n is odd. Define the function $f : V(G) \rightarrow \{1, 2, \dots, 2n\}$ as follows: $f(v_1) = 1, f(u_1) = 2, f(u_i) = 4i - 3$ for $2 \leq i \leq \lceil \frac{n}{2} \rceil$, $f(u_{n-i}) = 4i + 3$ for $0 \leq i \leq \lceil n/2 \rceil - 2$, $f(v_i) = 4(i - 1)$ for $2 \leq i \leq \lceil \frac{n}{2} \rceil$, $f(v_{n-i}) = 4(i + 2) - 2$ for $0 \leq i \leq \lceil \frac{n}{2} \rceil - 2$. For the above vertex labeling $f^*(E(G)) = \{1, 2, 3\}$. Then in this case maximum value of $f^*(E(G))$ is equal to 3. Case (ii) n is even. Define the function $f : V(G) \rightarrow \{1, 2, \dots, 2n\}$ as follows: $f(v_1) = 1, f(u_1) = 2, f(u_i) = 4i - 3$ for $2 \leq i \leq \frac{n}{2}$, $f(u_{n-i}) = 4i + 3$ for $0 \leq i \leq \frac{n}{2} - 1$, $f(v_i) = 4(i - 1)$ for $2 \leq i \leq \frac{n}{2} + 1$, $f(v_{n-i}) = 4(i + 2) - 2$ for $0 \leq i \leq \frac{n}{2} - 2$. For the above labeling we get $f^*(E(G)) = \{1, 2, 3\}$. Then in this case maximum value of $f^*(E(G))$ is equal to 3. By case (i) and (ii) the maximum value of the quotient labeling is 3. Then $q_l(f^*(E(G))) = 3$. But in G , $\delta(G) = 2$ and $\Delta(G) = 4$. Therefore $q_l(f^*)$ can take the values 3 or 4 or 5. Here $q_l(f^*) = 3$ and it is minimum. Hence $Q_L(M(C_n)) = 3$. $\square$

Theorem 6. *The quotient labeling number of $D_2(P_n)$ is 3.*

Proof. Consider two copies of P_n . Let $v_1, v_2, \dots, v_n$ be the n

vertices of rst copy of P_n and $u_1, u_2, \dots, u_n$ be the n vertices of second copy of P_n . Let $G = D_2(P_n)$. Now G has $2n$ vertices with $\deg(v_1) = \deg(v_n) = \deg(u_1) = \deg(u_n) = 2$ and $\deg(v_i) = \deg(u_i) = 4$ for $2 \leq i \leq n-1$. Define the function $f : V(G) \rightarrow \{1, 2, \dots, 2n\}$ as follows: $f(v_i) = 2i - 1$ for $1 \leq i \leq n$. $f(u_1) = 4$, $f(u_2) = 2$, $f(u_i) = 2i$ for $3 \leq i \leq n$. For the above labeling $f^*(E(G)) = \{1, 2, 3\}$. Then the maximum value of $f^*(E(G))$ is equal to 3. But in G , $\delta(G) = 2$ and $\Delta(G) = 4$. Therefore $q_l(f^*)$ can take the values 3 or 4 or 5. Here $q_l(f^*) = 3$ and it is minimum. Hence $Q_L(D_2(P_n)) = 3$. $\square$

Theorem 7. *The quotient labeling number of $D_2(C_n)$ is 5.*

Proof. Consider two copies of C_n . Let $v_1, v_2, \dots, v_n$ be the n vertices of rst copy of C_n and $u_1, u_2, \dots, u_n$ be the n vertices of second copy of C_n . Let $G = D_2(C_n)$. Now the graph G has $2n$ vertices with $\deg(v_i) = \deg(u_i) = 4$ for $1 \leq i \leq n$. We prove this theorem on two different cases of n . Case (i) n is odd. Define the function $f : V(G) \rightarrow \{1, 2, \dots, 2n\}$ as follows: $f(v_i) = i$ for $i = 1, 2$. $f(v_n) = 3$, $f(u_2) = 4$, $f(u_1) = 6$. $f(u_i) = 4(i-1) - 1$ for $3 \leq i \leq \lceil \frac{n}{2} \rceil$. $f(u_{n-i}) = 4(i+1) + 1$ for $0 \leq i \leq \lceil \frac{n}{2} \rceil - 2$. $f(v_i) = 4(i-1)$ for $3 \leq i \leq \lceil \frac{n}{2} \rceil$. $f(v_{n-i}) = 4(i+1) + 2$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil - 2$. For the above labeling $f^*(E(G)) = \{1, 2, 3, 4, 5\}$. Then in this case the maximum value of $f^*(E(G))$ is equal to 5. Case (ii) n is even. Define the function $f : V(G) \rightarrow \{1, 2, \dots, 2n\}$ as follows: $f(v_i) = i$ for $i = 1, 2$. $f(v_n) = 3$, $f(u_1) = 6$, $f(u_2) = 4$. $f(u_i) = 4(i-1) - 1$ for $3 \leq i \leq \frac{n}{2} + 1$, $f(u_{n-i}) = 4(i+1) + 1$ for $0 \leq i \leq \frac{n}{2} - 2$. $f(v_i) = 4(i-1)$ for $3 \leq i \leq \frac{n}{2} + 1$. $f(v_{n-i}) = 4(i+2) - 2$ for $1 \leq i \leq \frac{n}{2} - 2$. For the above labeling $f^*(E(G)) = \{1, 2, 3, 4, 5\}$. Then in this case the maximum value of $f^*(E(G))$ is equal to 5. By case (i) and (ii) the maximum value of the quotient labeling is 5. Then $q_l(f^*) = 5$. But in G , $\delta(G) = \Delta(G) = 4$. Therefore $q_l(f^*)$ can take the only value 5. Here $q_l(f^*) = 5$ and is minimum. Hence $Q_L(D_2(C_n)) = 5$. $\square$

Theorem 8. *The Quotient labeling number of a type of cater-*

pillar graph obtained from a path P_n by attaching i pendent vertices corresponds to every v_i , $1 \leq i \leq n$ of P_n by an edge is 2.

Proof. Let G be the graph obtained from a path P_n by attaching i pendent vertices corresponding to every v_i , $1 \leq i \leq n$ of P_n by an edge. Now G has $\frac{(n^2 + 3n)}{2}$ vertices and $\frac{(n^2 + 3n - 2)}{2}$ edges with $\deg(v_i) = i + 1$ for $i = 1$ and n . and $\deg(v_i) = i + 2$ for $2 \leq i \leq n - 1$. Let $v_1, v_2, \dots, v_n$ be the n vertices of P_n and let $v_{i,j}$ be the i pendent vertices attached with each v_i , $1 \leq i \leq n$ and $1 \leq j \leq i$. Define the function $f : V(G) \rightarrow \{1, 2, \dots, \frac{(n^2 + 3n)}{2}\}$ as follows: $f(v_{1,1}) = 1, f(v_1) = 2, f(v_2) = 3, f(v_i) = f(v_{i-1}) + i$ for $3 \leq i \leq n$. $f(v_{i,j}) = f(v_i) + j$ for $2 \leq i \leq n$ and $1 \leq j \leq i$. For the above labeling $f^*(E(G)) = \{1, 2\}$. Then the maximum value of $f^*(E(G))$ is equal to 2. But in G , $\delta(G) = 1$ and $\Delta(G) = n + 1$. Therefore $q_l(f^*)$ can take the values 2 or 3 or 4 or 5 or ... or $n + 2$. Here $q_l(f^*) = 2$ and is minimum. Hence $Q_L(G) = 2$. $\square$

Theorem 9. The Quotient labeling number of a type of caterpillar graph obtained from a path P_n by attaching r pendent vertices corresponds to every v_i , $1 \leq i \leq n$ and $i + (i + 2)^2 \leq r \leq i + (i + 3)^2$ of P_n by an edge is $3 + s$ and $s \geq -1$.

Proof. Let G be the graph obtained from a path P_n by attaching r pendent vertices corresponding to every v_i , $1 \leq i \leq n$ and $i + (i + 2)^2 \leq r \leq i + (i + 3)^2$ of P_n by an edge. Now G has $n(r + 1)$ vertices and $n(r + 1) - 1$ edges with $\deg(v_i) = r + 1$ for $i = 1$ and n . and $\deg(v_i) = r + 2$ for $2 \leq i \leq n - 1$. Let $v_1, v_2, \dots, v_n$ be the n vertices of P_n and let $v_{i,j}$ be the r pendent vertices attached with each v_i , $1 \leq i \leq n$, $i + (i + 2)^2 \leq r \leq i + (i + 3)^2$ and $1 \leq j \leq r$. There exists an integer s such that $s \geq -1$ and $i + (i + 2)^2 \leq r \leq i + (i + 3)^2$ and $1 \leq i \leq n$. Define the function $f : V(G) \rightarrow \{1, 2, \dots, n(r + 1)\}$ as follows: $f(v_{1,j}) = j$ for $1 \leq j \leq 3 + s - 1$, $f(v_{1,j}) = j + 1$ for $3 + s \leq j \leq r$, $f(v_1) = 3 + s$, $f(v_i) = f(v_{i-1,r}) + 1$ for $2 \leq i \leq n$. $f(v_{i,j}) = 3 + s + 4i - 5 + j$ for $2 \leq i \leq n$ and $1 \leq j \leq r$ and $s \geq -1$. For the above labeling $f^*(E(G)) = \{2, 3, \dots, 3 + s\}$ and $s \geq -1$. Then the maximum value of $f^*(E(G))$ is equal to $3 + s$ and $s \geq -1$. But in G , $\delta(G) = 1$ and $\Delta(G) = r + 2$. Therefore $q_l(f^*)$ can take the values 2 or 3 or 4 or ... or $r + 3$ for $i + (i + 2)^2 \leq r \leq i + (i + 3)^2$

and $1 \leq i \leq n$. Here $q_i(f^*) = 3 + s$ and $s \geq -1$ and it is minimum. Hence $Q_L(G) = 3 + s$ and $s \geq -1$. . $\square$

4 Conclusion

Quotient labeling number has been calculated for many family of graphs and our future work is to find the quotient labeling number for other new classes of graphs.

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