ON A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS
DEFINED BY CONVOLUTION AND INTEGRAL CONVOLUTION

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Abstract

In this paper, we introduce and study a subclass of harmonic univalent functions defined by convolution and integral convolution in the unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Also we obtain the coefficient bounds, extreme points, convex combination and convolution conditions.

Key words: Harmonic, analytic, univalent functions, convolution.

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1. Introduction

A continuous complex valued function $f = u + iv$ defined in a simply connected complex domain $D$ is said to be harmonic in $D$, if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain we can write $f = h + g$ where $h$ and $g$ are analytic in $D$. We call 'h' the analytic part and 'g' the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$.

Let $S_{H}$ denote the class of functions $f = h + g$ that are harmonic univalent and sense-preserving in the unit disk $U$ for which $f(0) = h(0) = f' - 1 = 0$. For $f = h + g \in S_{H}$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k$$

(1)

The class $K_{H}$ is defined as the subclass of $S_{H}$ consisting of all functions $f = h + g$ where $h$ and $g$ are given by

$$h(z) = z + \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1$$

(2)

The harmonic function $f = h + g$ for $g \equiv 0$ reduces to an analytic univalent function $f = h$. In 1984, Clunie and Sheil-Small [3] investigated the class $S_{H}$ and as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $S_{H}$ and its subclass such as Silverman and Silvia [10], Silverman [9], Avci and Zlotkiewicz [5], and Jahangiri [16] have studied the harmonic univalent functions.
The convolution of two functions of form
\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad F(z) = z + \sum_{k=2}^{\infty} A_k z^k \]
is defined as
\[ (f * F)(z) = f(z) * F(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k. \]  
(3)

The integral convolution is defined by
\[ (f \hat{*} F)(z) = z + \sum_{k=2}^{\infty} \frac{a_k A_k}{k} z^k. \]  
(4)

Recently W.G. Atshan et.al [11] and K.K. Dixit et.al [14] have defined and studied a subclass of harmonic univalent functions using integral convolution. They have studied the coefficient estimates, extreme points, convex combination, convolution, Bernardi and J-Kim-Srivastava operators.

Motivated by this aforementioned work, in the present paper we define a subclass \( \overline{B}_H(\beta, \alpha) \) of harmonic univalent functions and study the coefficient estimates, extreme points, convex combination, convolution conditions.

We define the subclass \( B_H(\beta, \alpha) \) of functions of the form (1) that satisfy the condition:
\[ \text{Re} \left[ \frac{\beta((h(z) * \phi(z)) - (g(z) * \psi(z)))}{(h(z) \hat{*} \phi(z)) + (g(z) \hat{*} \psi(z))} \right] > \alpha, \quad 0 < \alpha < \beta \leq 1, \]  
(5)

where \( \phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k \) and \( \psi(z) = \sum_{k=1}^{\infty} \mu_k z^k \) are analytic in \( U \) with the condition \( \lambda_k \geq 0, \ \mu_k \geq 0. \)

Let \( \overline{R}_H(\beta, \alpha) \) denote the subclass of \( \overline{B}_H(\beta, \alpha) \) consisting of functions \( f = h + g \in S_H \) such that \( h \) and \( g \) are of the form (2).

**Lemma (1) [17]:** Let \( \alpha \geq 0. \) Then \( \text{Re} \{ w \} > \alpha \) if and only if \( \|w - (1 + \alpha)\| < \|w + (1 - \alpha)\|, \) where \( \text{‘} w \text{’} \) be any complex number.

### 2. Coefficient Estimates:

We begin with a sufficient condition for function in the class \( \overline{B}_H(\beta, \alpha) \).

**Theorem (2.1):** Let \( f = h + g \) be defined in equation (1). If
\[ \sum_{k=2}^{\infty} \frac{\lambda_k (\alpha - k\beta)}{k(\beta - \alpha)} |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k (\alpha + k\beta)}{k(\beta - \alpha)} |b_k| \leq 1, \]  
(6)

where \( 0 < \alpha < \beta \leq 1, \) then \( f \) is harmonic univalent sense-preserving in \( U \) and \( f \in \overline{B}_H(\beta, \alpha). \)

**Proof:** For proving \( f \in \overline{B}_H(\beta, \alpha), \) we must show that (5) holds true. If
\[ W = \frac{\beta[h \hat{*} \phi] - (g * \psi)}{(h \hat{*} \phi) + (g \hat{*} \psi)} = \frac{A(z)}{B(z)}, \]
then by lemma (1), Re (w) ≥ α if and only if \[ |w - (1 + \alpha)| < |w + (1 - \alpha)| \).

It suffices to show that \[ |A(z) - (1 + \alpha) B(z)| - |A(z) + (1 - \alpha) B(z)| \leq 0 \]
where \[ A(z) = \beta \left\{ \left( h(z) * \phi(z) \right) - \left( g(z) * \psi(z) \right) \right\} \] and \[ B(z) = \left( h(z) \circ \phi(z) \right) + \left( g(z) \circ \psi(z) \right) \].

Consider
\[ |A(z) - (1 + \alpha) B(z)| \]

\[
= \beta \left[ z + \sum_{k=2}^{\infty} a_k \lambda_k z^k - \sum_{k=1}^{\infty} b_k \mu_k z^k \right] - (1 + \alpha) \left[ z + \sum_{k=2}^{\infty} \frac{a_k \lambda_k}{k} z^k + \sum_{k=1}^{\infty} \frac{b_k \mu_k}{k} z^k \right]
\]

\[
= \beta z - (1 + \alpha) z + \beta \sum_{k=2}^{\infty} a_k \lambda_k z^k - (1 + \alpha) \sum_{k=2}^{\infty} \frac{a_k \lambda_k}{k} z^k - \beta \sum_{k=1}^{\infty} b_k \mu_k z^k - (1 + \alpha) \sum_{k=1}^{\infty} \frac{b_k \mu_k}{k} z^k
\]

\[
= (\beta - 1 - \alpha) z + \sum_{k=2}^{\infty} \left( \beta \frac{1 + \alpha - k \beta}{k} \right) a_k \lambda_k z^k - \sum_{k=1}^{\infty} \left( \beta \frac{k \beta + 1 + \alpha}{k} \right) b_k \mu_k z^k
\]

\[
= (\alpha + 1 - \beta) z + \sum_{k=2}^{\infty} \left( \frac{1 + \alpha - k \beta}{k} \right) a_k \lambda_k z^k + \sum_{k=1}^{\infty} \left( \frac{1 + \alpha + k \beta}{k} \right) b_k \mu_k z^k
\]

\[
\leq (\alpha + 1 - \beta) |z| + \sum_{k=2}^{\infty} \frac{1 + \alpha - k \beta}{k} |a_k| |\lambda_k| |z|^k + \sum_{k=1}^{\infty} \left( \frac{1 + \alpha + k \beta}{k} \right) b_k |\mu_k| |z|^k
\]

\[ |A(z) + (1 - \alpha) B(z)| \]

\[
= \beta \left[ z + \sum_{k=2}^{\infty} a_k \lambda_k z^k - \sum_{k=1}^{\infty} b_k \mu_k z^k \right] + (1 - \alpha) \left[ z + \sum_{k=2}^{\infty} \frac{a_k \lambda_k}{k} z^k + \sum_{k=1}^{\infty} \frac{b_k \mu_k}{k} z^k \right]
\]

\[
= \beta z + (1 - \alpha) z + \sum_{k=2}^{\infty} \frac{k \beta + (1 - \alpha)}{k} a_k \lambda_k z^k - \sum_{k=1}^{\infty} \left( \frac{k \beta - (1 - \alpha)}{k} \right) b_k \mu_k z^k
\]

\[
= (\beta + 1 - \alpha) z - \sum_{k=2}^{\infty} \left( \frac{\alpha - 1 - k \beta}{k} \right) a_k \lambda_k z^k + \sum_{k=1}^{\infty} \left( \frac{\alpha - 1 + k \beta}{k} \right) b_k \mu_k z^k
\]

\[
\geq (\beta + 1 - \alpha) |z| - \sum_{k=2}^{\infty} \left( \frac{\alpha - 1 - k \beta}{k} \right) |a_k| |\lambda_k| |z|^k - \sum_{k=1}^{\infty} \left( \frac{\alpha - 1 + k \beta}{k} \right) |b_k| |\mu_k| |z|^k
\]

\[ |A(z) - (1 + \alpha) B(z)| - |A(z) + (1 - \alpha) B(z)| \]
\[
\leq (\alpha+1-\beta) |z| + \sum_{k=2}^{\infty} \left( \frac{1+\alpha-k\beta}{k} \right) |a_k| |\lambda_k| |z|^k + \sum_{k=1}^{\infty} \left( \frac{1+\alpha+k\beta}{k} \right) |b_k| |\mu_k| |\bar{z}|^k \\
+ (\alpha-1-\beta) |z| + \sum_{k=2}^{\infty} \left( \frac{\alpha-1-k\beta}{k} \right) |a_k| |\lambda_k| |z|^k + \sum_{k=1}^{\infty} \left( \frac{\alpha-1+k\beta}{k} \right) |b_k| |\mu_k| |\bar{z}|^k \\
= 2(\alpha-\beta) |z| + \sum_{k=2}^{\infty} \frac{2(\alpha-k\beta)}{k} |a_k| |\lambda_k| |z|^k + \sum_{k=1}^{\infty} \frac{2(\alpha-k\beta)}{k} |b_k| |\mu_k| |\bar{z}|^k \\
= 2 \sum_{k=2}^{\infty} \frac{\lambda_k}{k} |a_k| |\lambda_k| |z|^k + 2 \sum_{k=1}^{\infty} \frac{\mu_k}{k} |b_k| |\mu_k| |\bar{z}|^k - 2(\beta-\alpha) |z| \\
= 2 \sum_{k=2}^{\infty} \frac{\lambda_k}{k} |a_k| + 2 \sum_{k=1}^{\infty} \frac{\mu_k}{k} |b_k| - 2(\beta-\alpha) \leq 0 \\
\Rightarrow \sum_{k=2}^{\infty} \frac{\lambda_k}{k} |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} |b_k| \leq (\beta-\alpha). \\
\]

So, we have

\[
\sum_{k=2}^{\infty} \frac{\lambda_k}{k} |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} |b_k| \leq (\beta-\alpha).
\]

**Theorem (2.2):** Let \( f = h + g \) with \( h \) and \( g \) are given by (2). Then \( f \in \overline{B}_\mu(\beta, \alpha) \) if and only if

\[
\sum_{k=2}^{\infty} \frac{\lambda_k}{k(\beta-\alpha)} |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{k(\beta-\alpha)} |b_k| \leq 1.
\]

**Proof:** From theorem (2.1) to prove the necessary part let us assume that \( f \in \overline{B}_\mu(\beta, \alpha) \) using (5), we get

\[
\text{Re} \left[ \frac{\beta (h(z) \ast \phi(z)) - (g(z) \ast \psi(z))}{(h(z) \ast \phi(z)) + (g(z) \ast \psi(z))} \right] > \alpha
\]

\[
\text{Re} \left[ \frac{\beta \left( \sum_{k=2}^{\infty} |a_k| |\lambda_k| z^k - \sum_{k=1}^{\infty} |b_k| |\mu_k| \bar{z}^k \right)}{\sum_{k=2}^{\infty} |a_k| \frac{\lambda_k}{k} z^k + \sum_{k=1}^{\infty} |b_k| \frac{\mu_k}{k} \bar{z}^k} \right] > \alpha
\]
\[
\text{Re} \left( \frac{\beta z + \beta \sum_{k=2}^{\infty} \lambda_k |a_k| z^k - \beta \sum_{k=1}^{\infty} \mu_k |b_k| \bar{z}^k}{z + \sum_{k=2}^{\infty} |a_k| \frac{\lambda_k}{k} z^k + \sum_{k=1}^{\infty} |b_k| \frac{\mu_k}{k} \bar{z}^k} - \alpha \right) > 0
\]

\[
\text{Re} \left( \frac{\beta z + \beta \sum_{k=2}^{\infty} \lambda_k |a_k| z^k - \beta \sum_{k=1}^{\infty} \mu_k |b_k| \bar{z}^k - \alpha \sum_{k=2}^{\infty} |a_k| \frac{\lambda_k}{k} z^k - \alpha \sum_{k=1}^{\infty} |b_k| \frac{\mu_k}{k} \bar{z}^k}{z + \sum_{k=2}^{\infty} |a_k| \frac{\lambda_k}{k} z^k + \sum_{k=1}^{\infty} |b_k| \frac{\mu_k}{k} \bar{z}^k} \right) > 0
\]

\[
\text{Re} \left( (\beta - \alpha)z + \sum_{k=2}^{\infty} \left( \frac{\beta - \alpha}{k} \right) |a_k| \lambda_k z^k - \sum_{k=1}^{\infty} \left( \frac{\beta + \alpha}{k} \right) |b_k| \mu_k \bar{z}^k \right) > 0
\]

\[
\text{Re} \left( (\beta - \alpha)z - \sum_{k=2}^{\infty} \left( \alpha - k\beta \right) |a_k| \lambda_k z^k - \sum_{k=1}^{\infty} \left( \alpha + k\beta \right) |b_k| \mu_k \bar{z}^k \right) > 0
\]

\[
\text{Re} \left( (\beta - \alpha)z - \sum_{k=2}^{\infty} \left( \alpha - k\beta \right) |a_k| \lambda_k z^k + \sum_{k=1}^{\infty} \left( \alpha + k\beta \right) |b_k| \mu_k \bar{z}^k \right) > 0
\]

If we choose 'z' to be real and let \( z \to 1^- \), we obtain the condition (6) and the proof is complete.

3. Extreme Points:
In the following theorem, we obtain the extreme points of the class \( \overline{R}_H(\beta, \alpha) \).

Theorem 3.1: Let \( f \) be given by (2). Then \( f \in \overline{R}_H(\beta, \alpha) \) if and only if \( f \) can be expressed as
\[
f(z) = \sum_{k=1}^{\infty} (\delta_k h_k(z) + \gamma_k g_k(z)),
\]
where
\[
h_k(z) = z, h_k(z) = z + \frac{k(\beta - \alpha)}{\lambda_k (\alpha - k\beta)} z^k, k = 2, 3, \ldots,
\]
\[
g_k(z) = z + \frac{k(\beta - \alpha)}{\mu_k (\alpha + k\beta)} \bar{z}^k, k = 1, 2, \ldots, \sum_{k=1}^{\infty} (\delta_k + \gamma_k) = 1, \delta_k \geq 0, \gamma_k \geq 0.
\]

The extreme points of \( \overline{R}_H(\beta, \alpha) \) are \( \{h_k\} \) and \( \{g_k\} \).
Proof: Assume that $f$ can be expressed by (7). Then, we have

$$f(z) = \sum_{k=1}^{\infty} (\delta_k h_k(z) + \gamma_k g_k(z))$$

$$= \sum_{k=1}^{\infty} \left[ \delta_k \left\{ z + \frac{k(\beta - \alpha)}{\lambda_k(\alpha - k\beta)} z^k \right\} + \gamma_k \left\{ z + \frac{k(\beta - \alpha)}{\mu_k(\alpha + k\beta)} z^k \right\} \right]$$

$$= \sum_{k=1}^{\infty} \delta_k z + \sum_{k=2}^{\infty} \delta_k \frac{k(\beta - \alpha)}{\lambda_k(\alpha - k\beta)} z^k + \sum_{k=1}^{\infty} \gamma_k z + \sum_{k=1}^{\infty} \gamma_k \frac{k(\beta - \alpha)}{\mu_k(\alpha + k\beta)} z^k$$

$$= \sum_{k=1}^{\infty} (\delta_k + \gamma_k) z + \sum_{k=2}^{\infty} \frac{k(\beta - \alpha)}{\lambda_k(\alpha - k\beta)} \delta_k z^k + \sum_{k=1}^{\infty} \frac{k(\beta - \alpha)}{\mu_k(\alpha + k\beta)} \gamma_k z^k$$

$$= z + \sum_{k=2}^{\infty} \frac{k(\beta - \alpha)}{\lambda_k(\alpha - k\beta)} \delta_k z^k + \sum_{k=1}^{\infty} \frac{k(\beta - \alpha)}{\mu_k(\alpha + k\beta)} \gamma_k z^k.$$

Therefore,

$$\sum_{k=2}^{\infty} \left( \frac{\lambda_k(\alpha - k\beta)}{k(\beta - \alpha)} \right) \left( \frac{\frac{k(\beta - \alpha)}{\lambda_k(\alpha - k\beta)}}{\frac{k(\beta - \alpha)}{\lambda_k(\alpha - k\beta)}} \right) \delta_k + \sum_{k=1}^{\infty} \left( \frac{\mu_k(\alpha + k\beta)}{\frac{k(\beta - \alpha)}{\mu_k(\alpha + k\beta)}} \right) \gamma_k$$

$$= \sum_{k=2}^{\infty} \delta_k + \sum_{k=1}^{\infty} \gamma_k$$

$$= \delta_1 + \sum_{k=2}^{\infty} \delta_k - \delta_1 + \sum_{k=1}^{\infty} \gamma_k$$

$$= \sum_{k=1}^{\infty} \delta_k + \sum_{k=1}^{\infty} \gamma_k - \delta_1 = \sum_{k=1}^{\infty} (\delta_k + \gamma_k) - \delta_1$$

$$= (1 - \delta_1) \leq (\beta - \alpha).$$

So, $f \in \overline{R_H}(\beta, \alpha)$.

Conversely, let $f \in \overline{R_H}(\beta, \alpha)$. Then

$$a_k \leq \frac{k(\beta - \alpha)}{\lambda_k(\alpha - k\beta)}, \quad b_k \leq \frac{k(\beta - \alpha)}{\mu_k(\alpha + k\beta)}$$

and $\delta_k = \frac{\lambda_k(\alpha - k\beta)}{k(\beta - \alpha)}$, $\gamma_k = \frac{\mu_k(\alpha + k\beta)}{k(\beta - \alpha)}$, $k = 2, 3, \ldots$, and $k = 1, 2, \ldots$, respectively.
We define

\[ \delta_1 = 1 - \sum_{k=2}^{\infty} \delta_k - \sum_{k=1}^{\infty} \gamma_k . \]

Then, note that \( 0 \leq \delta_k \leq 1 \) \((k = 2,3,\ldots)\), \( 0 \leq \gamma_k \leq 1 \) \((k = 1,2,\ldots)\). Hence,

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k \]

\[ = z + \sum_{k=2}^{\infty} \frac{k(\beta - \alpha)}{k_k (\alpha - k\beta)} \delta_k z^k + \sum_{k=1}^{\infty} \frac{k(\beta - \alpha)}{k_k (\alpha + k\beta)} \gamma_k z^k \]

\[ = z + \sum_{k=2}^{\infty} \left[ z + \frac{k(\beta - \alpha)}{k_k (\alpha - k\beta)} z^k \right] \delta_k z^k + \sum_{k=1}^{\infty} \left[ z + \frac{k(\beta - \alpha)}{k_k (\alpha + k\beta)} z^k \right] \gamma_k z^k \]

\[ = \left(1 - \sum_{k=2}^{\infty} \delta_k - \sum_{k=1}^{\infty} \gamma_k \right) z + \sum_{k=2}^{\infty} h_k(z) \delta_k + \sum_{k=1}^{\infty} g_k(z) \gamma_k \]

\[ = \delta_1 h_1(z) + \sum_{k=2}^{\infty} h_k(z) \delta_k + \sum_{k=1}^{\infty} g_k(z) \gamma_k \]

\[ = \sum_{k=1}^{\infty} \delta_k h_k(z) + \sum_{k=1}^{\infty} \gamma_k g_k(z) , \text{ and the proof is complete.} \]

### 4. Convex combination:

Now, we show \( \overline{R}_H(\beta, \alpha) \) is closed under convex combination of its members.

**Theorem 4.1**: The class \( \overline{R}_H(\beta, \alpha) \) is closed under convex combination .

**Proof**: For \( j = 1,2,3,\ldots, \) let \( f_j \in \overline{R}_H(\beta, \alpha) \) where \( f_j \) is given by

\[ f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k + \sum_{k=1}^{\infty} b_{k,j} z^k . \]
Then, by (6), we have

\[
\sum_{k=2}^{\infty} \frac{\lambda_k (\alpha-k\beta)}{k(\beta-\alpha)} |a_{k,j}| + \sum_{k=1}^{\infty} \frac{\mu_k (\alpha+k\beta)}{k(\beta-\alpha)} |b_{k,j}| \leq 1.
\] (8)

For \( \sum_{j=1}^{\infty} e_j = 1, 0 \leq e_j \leq 1 \), the convex combination of \( f_j \) may be written as

\[
\sum_{j=1}^{\infty} e_j f_j(z) = z + \sum_{k=2}^{\infty} \left[ \sum_{j=1}^{\infty} e_j |a_{k,j}| \right] z^k + \sum_{k=1}^{\infty} \left[ \sum_{j=1}^{\infty} e_j |b_{k,j}| \right] \overline{z}^k
\]

Then, by (8), we have

\[
\sum_{k=2}^{\infty} \frac{\lambda_k (\alpha-k\beta)}{k(\beta-\alpha)} \left[ \sum_{j=1}^{\infty} e_j |a_{k,j}| \right] + \sum_{k=1}^{\infty} \frac{\mu_k (\alpha+k\beta)}{k(\beta-\alpha)} \left[ \sum_{j=1}^{\infty} e_j |b_{k,j}| \right]
\]

\[
= \sum_{j=1}^{\infty} e_j \left[ \sum_{k=2}^{\infty} \frac{\lambda_k (\alpha-k\beta)}{k(\beta-\alpha)} |a_{k,j}| + \sum_{k=1}^{\infty} \frac{\mu_k (\alpha+k\beta)}{k(\beta-\alpha)} |b_{k,j}| \right]
\]

\[
\leq \sum_{j=1}^{\infty} e_j (1) = 1.
\]

Therefore, \( \sum_{j=1}^{\infty} e_j f_j(z) \in \overline{R}_H(\beta, \alpha) \).

5. Convolution (Hadamard Product):

Define the convolution of two harmonic functions of the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \overline{z}^k \quad \text{and} \quad F(z) = z + \sum_{k=2}^{\infty} c_k z^k + \sum_{k=1}^{\infty} d_k \overline{z}^k.
\]

We define the convolution of two harmonic functions \( f \) and \( F \) as

\[
(f \ast F)(z) = f(z) \ast F(z) = z + \sum_{k=2}^{\infty} a_k c_k z^k + \sum_{k=1}^{\infty} b_k d_k \overline{z}^k.
\]

**Theorem (5.1):** For \( 0 \leq \tau \leq \alpha \leq 1 \), let \( f \in \overline{R}_H(\beta, \alpha) \) and \( F \in \overline{R}_H(\beta, \tau) \). Then \( f \ast F \in \overline{R}_H(\beta, \alpha) \).

**Proof:** Since \( f \in \overline{R}_H(\beta, \alpha) \) and \( F \in \overline{R}_H(\beta, \tau) \), then by theorem (2.2), we have
\[
\sum_{k=2}^{\infty} \frac{\lambda_k(\alpha-k\beta)}{k(\beta-\alpha)} |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k(\alpha+k\beta)}{k(\beta-\alpha)} |b_k| \leq 1
\] (9)

and

\[
\sum_{k=2}^{\infty} \frac{\lambda_k(\tau-k\beta)}{k(\beta-\tau)} |c_k| + \sum_{k=1}^{\infty} \frac{\mu_k(\tau+k\beta)}{k(\beta-\tau)} |d_k| \leq 1
\] (10)

From (10), we get the following inequalities

\[
c_k < \frac{k(\beta-\alpha)}{\lambda_k(\tau-k\beta)}, d_k < \frac{k(\beta-\alpha)}{\mu_k(\tau+k\beta)}.
\]

Therefore,

\[
\sum_{k=2}^{\infty} \frac{\lambda_k(\alpha-k\beta)}{k(\beta-\alpha)} |a_k||c_k| + \sum_{k=1}^{\infty} \frac{\mu_k(\alpha+k\beta)}{k(\beta-\alpha)} |b_k||d_k| \\
\leq \sum_{k=2}^{\infty} \frac{\lambda_k(\alpha-k\beta)}{k(\beta-\alpha)} |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k(\alpha+k\beta)}{k(\beta-\alpha)} |b_k| \leq 1.
\]

Then \( f \ast F \in \overline{R_\alpha(\beta)} \subset \overline{R_\beta(\alpha)} \), and the proof is complete.

References:


