Abstract: The objective of this article is the study of the dimensional numbers of sets of points when expressed as continued fractions; they observe some simple restriction as to their partial quotients. These sets of numbers obey the conditions on their partial quotients are often fractals. The middle third cantor set is considered, which consists of the numbers contain in the interval (0,1) and which has a base-3 expansion, whose continued fraction expansion involves only partial quotients. They are fractal subsets {0,1} in the middle third cantor fractal. The Hausdorff dimension is obtained which lies between 0 and 1.

Keywords: Continued fractions, The middle third cantor set, The Hausdorff dimension, The partial quotients.

1. Introduction
Fractals are represented by the geometric patterns and they have non-integer dimensions. The term Fractal was coined by Benoit Mandelbrot for the description of the irregular shapes, for example the coastlines and clouds. The important features of the fractal objects is the existence of the self similarity, means that a small part of the object resembles a whole part [1]. The Cantor set was discovered by Henry John Stephen Smith in 1874 and first introduced by German mathematician George Cantor in 1883. The Cantor set plays a very important role in many branches of mathematics. Mathematicians have been performing around with the continued fractions for a long time [3-5]. The idea is to express a number as a fraction with a denominator that itself is a fraction and this process continues. Rational numbers lead to a finite continued fractions and irrational numbers to infinite continued fractions. The continued fractions can be used to prove the irrationality of such numbers [2]. The German mathematician Lambert have used the concept of the continued fraction to prove the irrationality of the number \( \pi \) [11]. The Lagrange have used in solving the Diophantine equation [6].

In this paper, in section 2, the construction of the middle third cantor set fractal is described. In section 3, the definition and explanation about the continued fraction is given. In section 4, the Hausdorff dimension of the middle third Cantor set is calculated and the continued fraction interpretation is discussed.

2. Construction of the Middle third Cantor set
The Cantor set is a subset of the unit interval set [0,1] with which some conditions. The Cantor set is an uncountable set of measure zero with many interesting properties in the field of fractal theory [2]. The middle third cantor set is the union of two similar copies of itself and scaled by three, hence the Cantor set has fractal dimension \( \log(2)/\log(3) \). The Cantor set \( C \) is a subset of a closed set [0,1], is constructed by the following procedure. In Level 0, remove the open interval (1/3, 2/3) to obtain Level1=[0,1/3]+[2/3,1]. Now for Level 2, remove the middle thirds of the both of those segments and we obtain [0,1/9]+[29,1/3]+[2/3,7/9]+[8/9,1]. Similarly Level n+1 is obtained form Level n by removing the open intervals from each of the closed intervals of Level n. Since the removed intervals are open each interval in level n is closed and hence the limit set is also closed [3-5].

![Figure 1: Middle third Cantor set, \( F_L \) is the left part and \( F_R \) is the right part](image)

The self similarity property of the fractal used to define the fractal and it leads to find dimension of the fractal object. Let \( F \) be a middle third cantor set and Let \( S_1, S_2: R \to R \) be given by

\[
S_1(x) = \frac{1}{3} x; \quad S_2(x) = \frac{1}{3} x + \frac{2}{3}
\]

Where \( S_1 \) and \( S_2 \) are the basic self-similarities of the middle third cantor set. So that \( F = S_1(F) \cup S_2(F) \), where \( F \) is an attractor of the iterated function system containing of the contractions \{ \( S_1, S_2 \) \}[4].

3. Continued Fraction
A Continued fraction is an iterative process for the representation of a non-integer number as the sum of the integer part and the reciprocal of another non-integer number, then expressing this non-integer number as the sum of the integer part and the another reciprocal and so on [9-11]. The continued fractions are of two types, one is the finite continued fraction which ends with finite steps and the other one is the infinite continued fraction it contains infinite steps. In this...
expression the integers $a_i$'s are called the coefficients of the continued fraction.[2]

The continued fractions are being used instead of defining the sets of base $m$ expansion. A non-integer number $x$ can be expressed as

$$x = a_0 + \frac{1}{x_1}$$

and if $x_1$ is a non-integer then it can be expressed as

$$x_1 = a_1 + \frac{1}{x_2}$$

where $a_0$ and $a_1$ are integers with $x \geq 1$, $x > 1$.

Hence $x$ becomes

$$x = a_0 + \frac{1}{a_1 + \frac{1}{x_2}}$$

and proceeding like this

upto $m^{th}$ level with $x_m$ is an integer, we obtain,

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_{m-1} + \frac{1}{x_m}}}}}$$

Where the sequence of integers $a_0, a_1, a_2, a_3, \ldots$ are called the partial quotients of $x$, and the continued fraction expansion of $x$ is expressed as

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}$$

And the $m^{th}$ convergent is denoted by

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots + a_m}}}$$

These expressions of numbers can be convert into a quadratic equations by,

$$x = a_0 + \frac{1}{a_1 + \frac{1}{x}}$$

$$\Rightarrow a_0x^2 - a_0a_1x - a_0 = 0$$

4.2. The Continued Fraction

Every irrational number in the middle third cantor set, that is belongs to $[0,1)$ can be expressed as a continued fraction in the form

$$\sum_{n=0}^{\infty} \frac{(-1)^{a_n}}{a_n}$$

where $a_n \in N$, and $k \in N$

The typical set of numbers in $[0,1)$, all of whose partial quotients are 1 or 2 is considered. An important object in the theory of Continued fraction is the cantor set $F(n) = \{1, a_1, a_2, a_3, \ldots \} a_i \leq n \}$ where $n$ is a positive integer. Let the Cantor set $F$ be the set of positive real numbers $x$ with partial fraction expressions with all partial quotients equal to 1 or 2. Hence it is obvious that $F$ is closed and bounded [12]. Then $F$ is a fractal with Hausdorff dimension lies between 0 and 1. The set $F$ is a fractal with $x = 1 + 1/y$ or $x = 2 + 1/y$ with $y$ is a real number in $F$. Let the smallest number in $F$ be $c$, where $c = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}$. From this we obtain the quadratic equation as; 

$$c = 1 + \frac{1}{2 + \frac{1}{c}}$$

$$\Rightarrow 2c^2 - 2c - 1 = 0$$

Now solving this equation for the positive solution, we obtain,

$$c = \frac{1}{2}(1 + \sqrt{3})$$

And Let the largest number in $F$ be $d$, where

$$d = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}$$

From this we obtain the quadratic equation as;

Further the smallest and largest numbers can be found by solving of the above type quadratic equations and relation between them can be established.

4. The Continued Fraction interpretation and the Hausdorff Dimension

4.1. The Hausdorff Dimension

The Hausdorff dimension of a set is the limiting value of the fraction $\log N/\log M$, where $N$ is the number of intervals and $M$ is the scale or length of the intervals. For the middle third cantor set, it is well known that there are $2^n$ intervals, each of length $3^{n}$. It is clear that the dimension of the cantor set is $\log 2/\log 3$. Let $p_0, p_1, p_2, \ldots, p_{m-1}$ be the set of numbers with base-$m$ expansions with

$$\sum_{i=0}^{m-1} p_i = 1$$

where $0 \leq p_i \leq 1$. The Hausdorff dimension of the set of numbers $F = \{p_0, p_1, p_2, \ldots, p_{m-1}\}$ is defined by [1].

$$\dim F = -\frac{1}{\log m} \sum_{i=0}^{m-1} p_i \log p_i$$

$$\dim F = -\frac{1}{\log m} \sum_{i=0}^{m-1} p_i \log p_i$$

$$= -\frac{1}{\log 3} \sum_{i=0}^{3} p_i \log p_i$$

Since $m = 3$

$$= -\frac{1}{\log 3} [p_0 \log p_0 + p_1 \log p_1 + p_2 \log p_2]$$

$$= -\frac{1}{\log 3} [0 \log 0 + \frac{1}{3} \log \left(\frac{1}{3}\right) + \frac{2}{3} \log \left(\frac{2}{3}\right)]$$

By usual convention, $0 \log 0 = 0$.

$$= 0.5794 \approx 0.58$$

$$d = 2 + \frac{1}{1 + \frac{1}{d}}$$

(4.2) $$\Rightarrow d^2 - 2d - 2 = 0$$

Now solving this equation for the positive solution, we obtain, $d = 1 + \sqrt{3}$

The relation between the values $c$ and $d$ is obtained as; $d = 2 + \frac{1}{c}$, where 2 is the product value of the partial quotients 1 and 2. Now $S_1(x) = 1 + 1/x$ and $S_2(x) = 2 + 1/x$, where $F$ is the attractor of the iterated function system $\{S_1, S_2\}$ in the sense of $F = S_1(F) \cup S_2(F)$, that is $F = \sum_{i=1}^{m} S_i(F)$, and $F$ is closed and bounded which means non-empty compact set.

The Hausdorff dimension is 0.5312 and this value lies between 0 and 1.

Consider the typical set of numbers in $[0,1)$, all of whose partial quotients are 2 or 3 is considered. An important object in the theory of Continued fraction is the cantor set $F(n) = \{2, a_1, a_2, a_3, \ldots \} a_i \leq n \}$ where $n$ is a positive integer. Let the Cantor set $F$ be the set of positive real numbers $x$ with partial fraction expressions with all partial quotients equal to 2 or 3. Hence it is obvious that $F$ is closed and bounded. Then $F$ is a fractal with Hausdorff dimension lies in (0, 1). The set $F$ is a fractal with $x = 2 + 1/y$ or $x = 3 + 1/y$ with $y$ is a real number in $F$. The smallest number in $F$ is $c$, say, where $c = 2 + \frac{1}{3 + \frac{1}{2 + \frac{1}{3 + \cdots}}}$, we see immediately that
Now solving this equation for the positive solution, we obtain, 
\[ d = 3 + \frac{1}{c} \]
The relation between the values \( c \) and \( d \) is obtained as; 
\[ d = 6 + \frac{1}{c} \]
where 6 is the product value of the partial quotients 2 and 3. Now \( S_i(x) = 2 + 1/x \) and \( S_3(x) = 3 + 1/x \), where \( F \) is the attractor of the iterated function system \( \{ S_1, S_2, S_3 \} \) in the sense of 
\[ F = S_1(F) \cup S_2(F) \cup S_3(F) \], that is and \( F \) is closed and bounded which means non-empty compact set.
The Hausdorff dimension is 0.337437 and this value lies between 0 and 1. In the similar way for the partial quotients 1 and 3, the smallest and largest numbers in the set \( F \) is calculated from the equations 
\[ 3c^2 - 3c - 1 = 0 \] and 
\[ d^2 - 3d - 3 = 0 \] respectively. Also the relation between the numbers, 
\[ d = 3 + \frac{1}{c} \] and the Hausdorff dimension 0.454483 are obtained. These values are given in the table.

<table>
<thead>
<tr>
<th>Sl.No.</th>
<th>The Partial Quotients</th>
<th>Smallest Number</th>
<th>Largest Number</th>
<th>Relation B/W The Numbers</th>
<th>Hausdorff Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2</td>
<td>( c = \frac{1}{2}(1 + \sqrt{3}) )</td>
<td>( d = 1 + \frac{1}{c} )</td>
<td>( d = 2 + \frac{1}{c} )</td>
<td>0.531281</td>
</tr>
<tr>
<td>2</td>
<td>2.3</td>
<td>( c = \frac{1}{3}(3 + \sqrt{15}) )</td>
<td>( d = 3 + \frac{1}{c} )</td>
<td>( d = 6 + \frac{1}{c} )</td>
<td>0.337437</td>
</tr>
<tr>
<td>3</td>
<td>1.3</td>
<td>( c = \frac{1}{3}(3 + \sqrt{21}) )</td>
<td>( d = 3 + \frac{1}{c} )</td>
<td>( d = 3 + \frac{1}{c} )</td>
<td>0.454483</td>
</tr>
</tbody>
</table>

The Hausdorff dimension values can be calculated for all the set of numbers with any pair of the partial quotients.

References
