STUDY OF APPROXIMATIONS USING EXPONENTIAL FUNCTIONS IN FUZZY SYSTEMS

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Abstract: A fuzzy system can uniformly approximate any real continuous function on a compact domain to any degree of accuracy. These outcomes can be considered as an existence of optimum fuzzy systems. Li-Xin Wang discussed a similar problem using Gaussian membership function and Stone-Weierstrass Theorem and established that fuzzy systems, with product inference, centroid defuzzification and Gaussian functions, are capable of approximating any real continuous function on a compact set to arbitrary accuracy. In this research review, a detailed study on similar approximation problem by using exponential membership functions has been done.

Keywords: Fuzzy systems, Stone-Weierstrass Theorem, approximations, fuzzification, defuzzification

I. Introduction
Fuzzy systems theory has numerous applications in various fields namely control systems, signal processing, image processing etc.. The foremost objective of these applications is to construct a fuzzy system to approximate the desired control or decision. Scientifically, this is to identify a mapping from the input space to the output space which can approximate the desired function within a given accuracy. Hence the problems of designing fuzzy systems can be viewed as approximation problems. Fuzzy system experts studied several approximation problems of fuzzy systems that are closely related with fuzzy basic functions.

A detailed discussion on fuzzy system approximation problem using Gaussian membership function is done by Wang [8]. He established that fuzzy systems are capable of approximating any real continuous function on a compact set to arbitrary accuracy. In this paper we study a similar approximation problem by using simple exponential membership functions.

Fuzzy Systems:
A universe of discourse W is an assortment of objects which can be distinct or continuous. A fuzzy set A in a universe of discourse W is characterized by a membership function \( \mu_A \): W \( \rightarrow \) [0, 1]. For example, if W is the set of all human beings, the concepts such as ‘child’, ‘young’, ‘old’ and ‘very old’ are called fuzzy concepts that are characterized by fuzzy sets. The fuzzy sets are characterized by the following principal elements. 1. Fuzzification Interface 2. Fuzzy Inference Machine (with Fuzzy Rule Base) 3. Defuzzification Interface

These three principal elements constitute a fuzzy system in W. The architecture of a fuzzy system is given below. Throughout this study, W \( \subset \) \( \mathbb{R}^n \) where R is the set of all real numbers. Since a multi-output system can be separated into a collection of single output systems, we shall consider multi-input-single output (MISO) fuzzy systems f: W \( \subset \) \( \mathbb{R}^n \) \( \rightarrow \) \( \mathbb{R}^n \).

Fuzzy Inference Machine

Fig 1 Architecture of Fuzzy Systems

From this, it can easily be observed that the number of fuzzy sets defined in the input universe of discourse and the specific membership functions are the important factors that are used to determine a fuzzification interface. We can view these two factors as design parameters of a fuzzification interface. Specifically, for a MISO fuzzy system, the design parameters of a fuzzification interface are

(1) \( m \), \( i=1,2,...,n \), the number of fuzzy sets defined in the subspace \( \{ x_i : x = (x_1,x_2,...,x_n) \in W \} \) of W corresponding to the \( i^{th} \) coordinates and

(2) \( \mu_{A_{i,j}} \) for \( i=1,2,...,n \), \( j=1,2,...,m \), the membership function of the \( j^{th} \) fuzzy set defined in the \( i^{th} \) subspace of W.

Fuzzy Rule Base: The fuzzy rule base is a set of well-defined linguistic labels in the form of “IF a set of conditions are satisfied, THEN a set of consequences are inferred”, where the conditions and the consequences are associated with fuzzy concepts. For example, in the case of an n-input –single output fuzzy system, the fuzzy rule base may consist of the following rules:

\( R_i : \) IF \( x_1 \) is \( A_{1,j} \) and \( x_2 \) is \( A_{2,j} \) and \( ... \), \( x_n \) is \( A_{n,j} \), THEN \( z \) is \( B^{j} \)

(1)

where \( x_1, x_2, ..., x_n \) are the inputs to the fuzzy system, \( z \) is the output of the fuzzy system, \( A_{i,j} \) and \( B^{j} \), for \( j=1,2,...,K \) are linguistic terms and K is the number of fuzzy rules in the fuzzy rule base. By relating each linguistic term in the fuzzy rules with a membership function, we specify the meaning of the fuzzy rules in determined fuzzy sense. There are many different kinds of fuzzy rules; see [4] for a complete discussion.

Fuzzy Inference Machine is a decision-making logic which employs fuzzy rules from the fuzzy rule based determine fuzzy outputs of a fuzzy system corresponding to the fuzzified inputs to the fuzzy system. It is the fuzzy inference machine that simulates a human decision-making procedure based on fuzzy concepts and linguistic statements. There are many different kinds of fuzzy logic which may be used in a comprehensive study on
Defuzzification Interface

defuzzifies the fuzzy output of a fuzzy system to generate non-fuzzy output. There are three existing defuzzification methods, namely: centroid, max-criterion and mean of maximum (see [4] for details). The design parameters of a defuzzification interface are: (1) number of fuzzy sets defined in the output universe of discourse R; (2) specific membership functions of these fuzzy sets; and (3) which defuzzification method is used.

In summary, a fuzzy system has the following design parameters:

- \( P_1 \) Number of fuzzy sets defined in the input and output universes of discourse;
- \( P_2 \) Membership functions of these fuzzy sets;
- \( P_3 \) Number of fuzzy rules in the fuzzy rule base;
- \( P_4 \) Linguistic statements of the fuzzy rules;
- \( P_5 \) Decision making logic used in the fuzzy inference machine; and,
- \( P_6 \) Defuzzification method.

The number of fuzzy sets defined in the input and output universes of discourse and the number of fuzzy rules in the fuzzy rule base heavily influence the complexity of a fuzzy system, where complexity includes time complexity and space complexity. These parameters can be viewed as structure parameters of a fuzzy system. In general, the larger these parameters are, the more complex is the fuzzy system, and the higher is the expected performance of the fuzzy system. Hence, there is always a trade off between complexity and accuracy in the choice of these parameters; and their choice is usually quite subjective.

The membership functions of the fuzzy sets are used by the fuzzy inference machine very important, and may be the most flexible component in the fuzzy system. A sophisticated fuzzy system may need a sophisticated fuzzy inference machine. The role of the defuzzification strategy in a fuzzy system is somewhat unclear because there are only three defuzzification methods available, among which the centroid method seems to provide the best performance for most application [4].

Let \( Z \) be a set of real continuous functions on \( W \). Assume \( Z \) is an algebra if \( Z \) is closed under addition, multiplication and scalar multiplication. \( Z \) separates points on \( W \) if for every \( x, y \in W \), \( x \neq y \), there exists \( f \in Z \) such that \( f(x) \neq f(y) \). \( Z \) vanishes at no point of \( W \) if for each \( x \in W \); there exists \( f \in Z \) such that \( f(x) \neq 0 \). We use the following theorem due to Stone and Weierstrass [6].

**Theorem 1.** The set of fuzzy systems with product inference, centroid defuzzification and exponential membership functions, denoted by \( Y \) in the sequel, consists of all functions

\[
f: W \subset \mathbb{R}^n \rightarrow \mathbb{R} \text{ defined by } f(x) = \frac{1}{\text{exp}^{-1}} \sum_{j=1}^{K} \left( z_j \prod_{i=1}^{n} \mu_{A_{(i,j)}}(x_i) \right),
\]

(3)

\( x = (x_1, x_2, ..., x_n) \in W \) where \( K \) is the number of fuzzy rules in the fuzzy rule base, \( \mu_{A_{(i,j)}}(x_i) \) is the exponential membership function in (2) and \( Z_j \) is the point in the output space \( R \) at which the fuzzy set \( A_{(i,j)} \) achieves its maximum membership value.

We assume \( K \geq 1 \), and that \( W \) is compact [5].

From (3), we observe that if we view the fuzzy inference machine and defuzzification interface as an integrated part, then product inference logic can be explained as that the ‘weight’ of Rule \( j \) to the contribution of determining the output of the fuzzy system for input \( x \) equals

\[
\prod_{i=1}^{n} \mu_{A_{(i,j)}}(x_i) \cdot \text{Centroid defuzzification means that the non-fuzzy output of the fuzzy system is a weighted sum of the } K \text{ points in } R \text{ at which the membership function characterizing the linguistics terms in the conclusion parts of the } K \text{ rules achieve their maximum values, where the ‘weights’ are determined by the product inference machine.}

The design parameters of the fuzzy systems in \( Y \) are:

- \( m_i, i=1,2,...,n \), the number of fuzzy sets defined in the \( i^{th} \) subspace of the input universe.
Let $d_e(f_1, f_2)$ be the sup-metric [5] defined by

$$d_e(f_1, f_2) = \text{Sup} \{ |f_1(x) - f_2(x)| : x \in W \}.$$ 

Then $(Y, d_e)$ is a metric space.

**Proposition 1:** $Y$ is non-empty.

**Proof:** Follows from the assumption $K \geq 1$.

**Proposition 2:** $(Y, d_e)$ is well-defined.

**Proof:** Since $Y$ is non-empty, we only need to prove that the denominators of (3) is nonzero for any $x \in W$. Based on (2), the exponential membership functions are nonzero; hence, the denominator of (3) is nonzero.

From the proof of Proposition 2, it can be understood that if the membership functions are changed into the triangular form, then the resulting $d_e$ may not be well-defined, because for an arbitrary $f$ in such $Y$, it cannot be guaranteed that the denominator of $f$ is non zero for every $x \in W$.

Next we use Theorem 1 to prove that $(Y, d_e)$ is dense in $C[W, d_e]$, where $C[W]$ is the set of all real continuous functions defined on the compact set $W$. In order to use Theorem 1, we need to show that $Y$ is an algebra, $Y$ separates points on $W$, and, $Y$ vanishes at no point of $W$.

**Proposition 3:** $(Y, d_e)$ is algebra.

**Proof:**

Let $f_1, f_2 \in Y$. Then $f_1(x) = \sum_{i=1}^{k_1} (z_{11}^{i} \prod_{i=1}^{n} \mu A_{1_{0_{1}}}(x_i))$ and $f_2(x) = \sum_{i=1}^{k_2} (z_{12}^{i} \prod_{i=1}^{n} \mu A_{2_{0_{2}}}(x_i))$.

$$f_1(x) + f_2(x) = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} (z_{11}^{i} + z_{12}^{j}) \prod_{i=1}^{n} \mu A_{1_{0_{1}}}(x_i) \mu A_{2_{0_{2}}}(x_i).$$

**Proposition 4:** $(Y, d_e)$ separates points on $W$.

**Proof:** We can prove this by constructing a required $f$. Let $x^0 \neq y^0$. We wish to construct $f$ such that $f(x^0) \neq f(y^0)$. Let $x^0 = (x_1^0, x_2^0, ..., x_n^0)$ and $y^0 = (y_1^0, y_2^0, ..., y_n^0)$. If $x_i^0 \neq y_i^0$.

$$f(x^0) = \frac{\sum_{i=1}^{k_1} \prod_{i=1}^{n} \mu A_{1_{0_{1}}}(x_i)}{\sum_{j=1}^{k_2} \prod_{i=1}^{n} \mu A_{2_{0_{2}}}(x_i)}.$$
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\[ f(y^0) = \frac{z_i^0 + z_j^0 y_j}{\delta + \gamma} = \left( \frac{\omega}{\delta + \gamma} \right) z_i^0 + \left( \frac{\gamma}{\delta + \gamma} \right) z_j^0. \]

We need to prove that \( f(x^0) \neq f(y^0) \), that is

\[ \left( \frac{\omega}{\alpha + \beta} \right) z_i^0 + \left( \frac{\gamma}{\alpha + \beta} \right) z_j^0 \neq \left( \frac{\omega}{\alpha + \beta} \right) z_i^0 + \left( \frac{\gamma}{\alpha + \beta} \right) z_j^0. \]

Claim: \( f(x^0) \neq f(y^0) \)

\[ f(x^0) = f(y^0) \iff \left( \frac{\omega}{\alpha + \beta} \right) = \left( \frac{\delta}{\delta + \gamma} \right) \text{ and } \left( \frac{\gamma}{\alpha + \beta} \right) = \left( \frac{\gamma}{\delta + \gamma} \right). \]

\[ \iff e^{-\gamma y_j} = \prod_{i=1}^{n} \frac{1}{y_i} \exp \left( - \frac{x_i^0 - y_i^0}{y_i} \right) \prod_{i=1}^{n} \frac{1}{y_i} \exp \left( - \frac{y_i^0}{x_i} \right). \]

\[ \iff e^{-2} = \prod_{i=1}^{n} \exp \left( - \frac{x_i^0 - y_i^0}{y_i} \right) \exp \left( - \frac{y_i^0}{x_i} \right) \]

\[ = \prod_{i=1}^{n} \exp \left( - \frac{(x_i^0)^2 + (y_i^0)^2}{x_i^0 y_i^0} \right). \]

\[ \iff e^{-2} = \exp \left( - \sum_{i=1}^{n} \frac{(x_i^0)^2 + (y_i^0)^2}{x_i^0 y_i^0} \right). \]

\[ \iff 2 = \sum_{i=1}^{n} \frac{(x_i^0)^2 + (y_i^0)^2}{x_i^0 y_i^0}. \]

This is not possible as \( x_i^0 \neq y_i^0 \).

Hence \( f(x^0) \neq f(y^0) \).

\[ \square \]

**Proposition 5:** \((Y, d_e)\) vanishes at no point on \( \text{W} \).

**Proof:** By observing (3) and (2), we simply choose all \( z_j^0 > 0 \) \((j = 1, 2, \ldots, k)\), i.e., any \( f \in \text{Y} \) with \( z_j^0 > 0 \) serves as the required \( f \).

The next theorem shows that the fuzzy systems in \( \text{Y} \) can approximate continuous functions.

**Theorem 2:** For any given real continuous function \( g \) on the compact set \( W \subset \mathbb{R}^n \) and arbitrary \( \varepsilon > 0 \), there exists \( f \in \text{Y} \) such that \( \sup \{ |f(x) - g(x)| : x \in W \} < \varepsilon \).

**Proof:** It is obvious that \( \text{Y} \) is a set of real continuous function on \( \text{W} \). The proof follows Theorem 1 and Propositions 3-5.

\[ \square \]