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EQUITABLY INDEPENDENT PARTITION OF A GRAPH

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Abstract:

In a democratic society, a remarkable and desirable feature of constitution should be for each person is equality of status, power, wealth and opportunity. Keeping this idea as a curcial condition, Sampathkumar has introduced the concept of equitable domination. On the other hand some social status like marital relationship must be enjoyed individually. Also the information and password security for financial matters must be independent in order to retain uniqueness. Such issues have motivated the concept of independent domination, which was formalized by Berge and Ore. The notation $i(G)$ was introduced by Cockayne and Hedetniemi [3]. A survey on independent domination can be found in Goddard and Henning. A natural question arise whether it is possible to think about partitioning people who are independent or of not nearly equal (varied) status: In attempt to answer this question, the idea of equitably independent partition is introduced.

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Key Words and Phrases: inequitable, Equitable independent partition, Equitable independent partition number.

1. Introduction

By a graph $G = (V, E)$ we mean a connected, finite, non-trivial, undirected graph with neither loops nor multiple edges. For graph theoretic terminology, we refer to Chartrand and Lesniak [1]. A *graph* G is a finite non-empty set of elements called *vertices* together with a set of unordered pairs of distinct vertices of G called *edges [7]*. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$ respectively. The number of vertices in G is called the *order* of G and the number of edges in G is called the *size* of G. A graph is *trivial* if its vertex set is a singleton set. A graph with no edges is called a *totally disconnected graph*. The *degree* of a vertex v in a graph G is defined to be the number of edges incident with v and is denoted by $d(v)$ or $\deg v$. A vertex of degree zero is an *isolated vertex* and a vertex of degree one is a *pendent vertex*. An edge incident with a pendent vertex is called a *pendent edge*. The minimum of $\{\deg v \mid v \in V(G)\}$ is denoted by δ and the maximum of $\{\deg v \mid v \in V(G)\}$ is denoted by Δ. A graph G is said to be regular if every vertex in G has the same degree. If every vertex in G has degree r then it is called $r-$regular. If every vertex in G has degree r or $r+1$ then it is called a bi-regular graph. A graph G is *complete* if every pair of distinct vertices of G is adjacent in G. A complete graph on n vertices is denoted by K_n. A set $S \subseteq V(G)$ of a graph G is called an *independent set* if no two vertices of S are adjacent in G. The maximum cardinality of an independent set is called the *independence number* of G and is denoted by $\beta_0(G)$. A *proper coloring* of a graph G is an assignment of colors to the vertices of G, one color to each vertex, in such a way that no two adjacent vertices receive the same color. A coloring in which k colors are used is called a k*-coloring*. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum integer k with G admits a k-coloring. If C is a k-coloring of a graph G and if V_i denotes the set of vertices of G that receive the color i, then the coloring C is denoted by $C = \{V_1, V_2, \cdots, V_k\}$. Here $V_i^{'}s$ are called the *color classes* of C and this coloring C is also called a *chromatic partition* of G. For $n \geq 2$, the graph $K_{1,n}$ is called a *star* and here the vertex of degree n is called the central vertex of $K_{1,n}$. The graph G obtained from $K_{1,r}$ and $K_{1,s}$ by joining their central vertices by an edge is called a bi-star and is denoted by $B(r,s)$. A *path* of length n in a graph G, denoted by P_n, is a sequence $(u_0, u_1, ..., u_n)$ of distinct vertices such that the vertices u_i and u_{i+1} are adjacent for each i with $1 \leq i \leq n-1$. A *cycle* of length n in a graph G, denoted by C_n, is a sequence $(u_0, u_1, ..., u_{n-1}, u_0)$ of vertices of G, such that, for $1 \leq i \leq n-2$, the vertices u_i and u_{i+1} are adjacent, u_{n-1} and u_0 are adjacent and $u_0, u_1, ..., u_{n-1}$ are distinct. A cycle C_n of length n is called even or odd according as n is *even or odd*. For $n \geq 4$, the wheel on n vertices, denoted by W_n is defined to be the graph $K_1 + C_{n-1}$. That is, a wheel W_n

is obtained from a cycle C_{n-1} by adding a vertex, say $v,$ and joining it to all the vertices of the cycle $C_{n-1}.$

2. Equitably independent partition

Definition 2.1: Two vertices v_i and v_j are said to be ***inequitable*** if $|d(v_i)-d(v_j)|\geq 2.$

Definition 2.2: A partition $\pi=\{V_1,V_2,\ldots V_k)$ is called an ***equitably independent partition*** of $V(G)$ if each V_i is either independent or if two vertices in V_i are adjacent then they are inequitable.

The ***equitably independent partition number*** $\pi_{ei}(G)$ is equal to the minimum k such that G has a equitably independent partition of size k .

Existence 2.3: Let $V(G)=\{v_1,v_2,\ldots,v_n\}.$ Since every singleton set is an independent set, the partition $\{\{v_1\},\{v_2\},\ldots,\{v_n\}\}$ is an equitably independent partition. Thus, every graph G has an equitably independent partition and hence equitably independent partition number $\pi_{ei}(G)$ is well defined.

Observation 2.4: For any graph $G,1\leq\pi_{ei}(G)\leq n-\beta_0+1.$

Proof: If $V(G)$ itself is an independent set, then $\pi_{ei}(G)=1.$ If not, let $\{v_1,v_2,\ldots,v_{\beta_0}\}$ be an independent set of $V(G).$ Clearly, $\{\{v_1,v_2,\ldots,v_{\beta_0}\},\{v_{\beta_0+1}\},\ldots,\{v_n\}\}$ is an equitably independent partition of $G.$ Hence $\pi_{ei}(G)\leq n-\beta_0-1.$

Theorem 2.5: For a complete bipartite graph $K_{m,n}$, $\pi_{ei}(K_{m,n})=\begin{cases}2 & if\ |m-n|\leq 1\\ 1 & if\ |m-n|\geq 2\end{cases}$

for every $m,n\geq 1.$

Proof: Let (V_1,V_2) be the partition of the complete bipartite graph $K_{m,n}$ with $|V_1|=m$ and $|V_2|=n$. Hence for every vertex $u\in V(K_{m,n}),\ d(u)=\begin{cases}n & if\ u\in V_1\\ m & if\ u\in V_2\end{cases}$

Since $K_{m,n}$ is a bipartite graph, V_1 and V_2 are independent sets of $V(K_{m,n}).$ Hence $\pi_{ei}(K_{m,n})\leq 2.$ If $|m-n|\leq 1,$ then every two vertices which are adjacent in $K_{m,n}$ is equitable. Hence no two vertices in $K_{m,n}$ can be in the same partition and hence

$\pi_{ei}\left(K_{m,n}\right)=2$. If $|m-n|\geq 2$, then every two vertices which are adjacent in $K_{m,n}$ is inequitable. Hence $V(K_{m,n})$ is an equitably independent partition. Hence $\pi_{ei}\left(K_{m,n}\right)=1$.

Corollary 2.6: For a star graph $K_{1,n}$ with $n+1$vertices, $\pi_{ei}(K_{1,n})=1,\ n\geq 3$.

Theorem 2.7: For a bi-star graph $B(n_1,n_2)$ with n_1+n_2+2 vertices,

$$\pi_{ei}\left(B(n_1,n_2)\right)=\begin{cases}2 & if\ |n_2-n_1|\leq 1\\ 1 & if\ |n_2-n_1|\geq 2\end{cases}$$

Proof: Let $V(B(n_1,n_2))=\{u,u_1,u_2,\ldots,u_{n_1},v,v_1,v_2,\ldots,v_{n_2}\}$. Here $\{u,v_1,\ldots,v_{n_2}\}$ and $\{v,u_1,\ldots u_{n_1}\}$ are independent sets of $V(B(n_1,n_2))$. Hence $\pi_{ei}\left(B(n_1,n_2)\right)\leq 2$. If $|n_2-n_1|\leq 1$, then $|d(u)-d(v)|\leq 1$. Hence the vertices u and v do not belong to the same partition. Hence $\pi_{ei}\left(B(n_1,n_2)\right)=2$. If $|n_2-n_1|\geq 2$, then $|d(u)-d(v)|\geq 2$. Hence the vertices u and v belong to the same partition. Hence $\pi_{ei}\left(B(n_1,n_2)\right)=1$.

Theorem 2.8: For any graph G, $\pi_{ei}(G)\leq \chi(G)$.

Proof: Let $\pi=\{V_1,V_2,...,V_\chi\}$ be any χ coloring of G. Since each V_i, $1\leq i\leq \chi$ is an independent set, $\pi=\{V_1,V_2,...,V_\chi\}$ is an equitably independent partition of G. Hence $\pi_{ei}(G)\leq \chi(G)$.

Corollary 2.9: If G is a tree, then $\pi_{ei}(G)\leq 2$.

For $\pi_{ei}(G)=1$ and $\pi_{ei}(G)=2$ the graph is given in Figure 1.a. and 1.b.

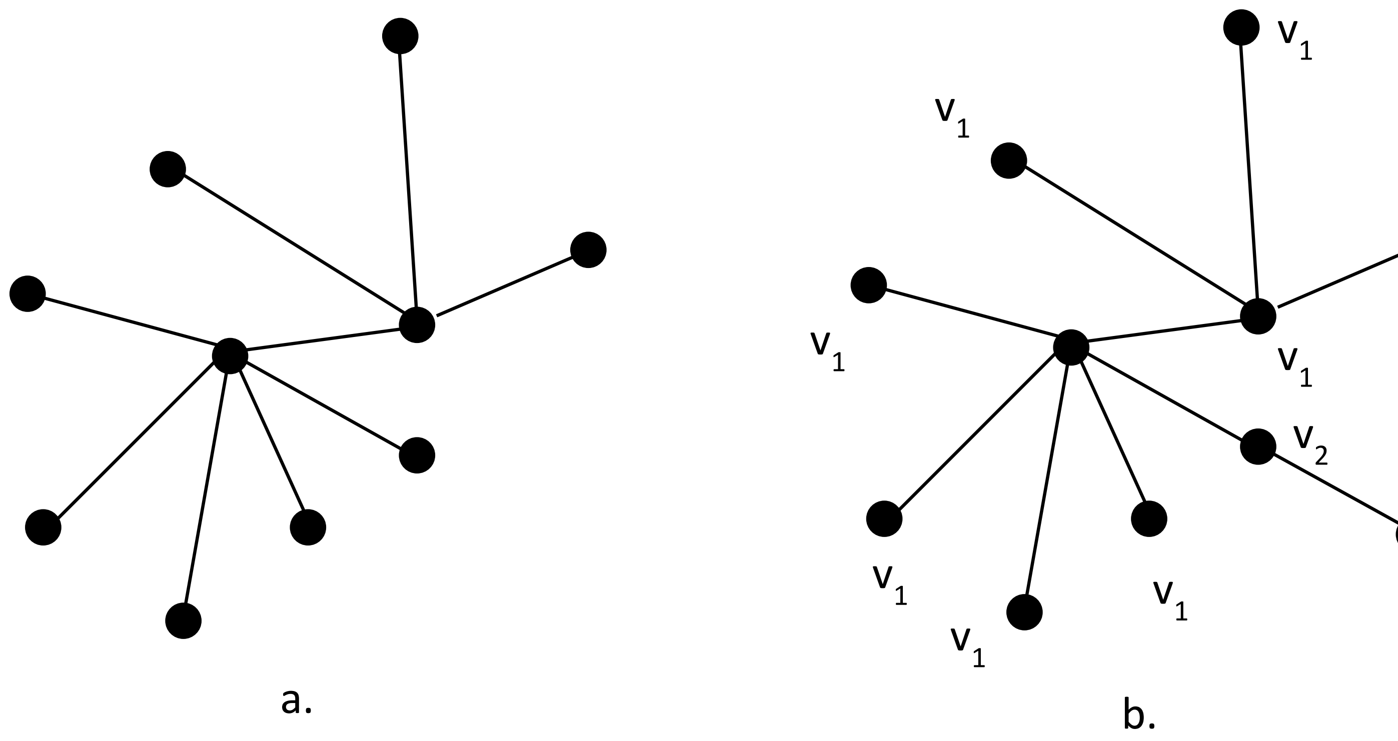

Figure 1 : a. A graph G with $\pi_{ei}(G) = 1$. b. A graph G with $\pi_{ei}(G) = 2$.

Theorem 2.10: If G is a regular graph or a bi-regular graph with n vertices, then $\pi_{ei}(G) = \chi(G)$.

Proof: Let $V(G) = \{v_1, v_2, \ldots v_n\}$. If G is a regular graph, then $|d(v_i) - d(v_j)| = 0$ for all $v_i, v_j \in V(G)$. If G is a bi-regular graph, then $|d(v_i) - d(v_j)| \leq 1$ for all $v_i, v_j \in V(G)$. Hence in both the cases, no two adjacent vertices belong to the same set in any equitably independent partition. Thus every set is an independent set in any equitably independent partition. Hence $\pi_{ei}(G) = \chi(G)$.

Corollary 2.11: If G is a complete graph with n vertices, then $\pi_{ei}(G) = n$.

Proof: Since every complete graph is regular and the number of independent set for a complete graph with n vertices is n, $\pi_{ei}(G) = n$.

Corollary 2.12: For a cycle C_n with n vertices, $\pi_{ei}(C_n) = \begin{cases} 3 & \textit{if } n \textit{ is odd} \\ 2 & \textit{if } n \textit{ is even} \end{cases}$

Proof: A cycle C_n with n vertices is a 2-regular graph. Hence $\pi_{ei}(C_n) = \chi(C_n)$. Since for a cycle C_n with n vertices, $\chi(C_n) = \begin{cases} 2 & if\ n\ is\ even \\ 3 & if\ n\ is\ odd \end{cases}$ for $n \geq 3$. we have,

$$\pi_{ei}(G) = \begin{cases} 3 & if\ n\ is\ odd \\ 2 & if\ n\ is\ even \end{cases}.$$

Corollary 2.13: For a path P_n with n vertices, $\pi_{ei}(P_n) = 2$ for all $n \geq 2$.

Proof: A path $P_n = (v_1, v_2, \ldots, v_{n-1}, v_n)$ with n vertices is a bi-regular graph with $d(v_i) = 2$ for all $2 \leq i \leq n-1$ and $d(v_1) = 1 = d(v_n)$. Hence $\pi_{ei}(P_n) = \chi(P_n)$. Since $\chi(P_n) = 2$, $\pi_{ei}(P_n) = 2$ for all $n \geq 2$.

Theorem 2.14: For a wheel graph W_n with $n \geq 4$ vertices,

$$\pi_{ei}(W_n) = \begin{cases} 3 & if\ n = 5 \\ 3 & if\ n\ is\ even \\ 2 & if\ n\ is\ odd \end{cases}$$

Proof: Let $V(W_n) = \{v_1, v_2, \ldots, v_n\}$. Let v_1 be the central vertex and $C = (v_2, v_3, ..., v_n, v_2)$ be the cycle of the wheel graph W_n. Here $d(v_1) = n-1$ and $d(v_i) = 3$, for all i, $2 \leq i \leq n.$

Case 1: $n = 5$.

Here $|d(v_i) - d(v_j)| \leq 1$ for all $v_i, v_j \in V(W_n)$. Hence no two adjacent vertices belong to the same partition. As the central vertex v_1 is adjacent to every other vertex and the 4-cycle has 2 independent sets, the partition $\{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}$ is an equitably independent partition. That is, $\pi_{ei}(W_n) = 3$.

Case 2: $n \geq 6$.

As the central vertex v_1 is adjacent to every other vertex $v_2, v_3, \ldots, v_n$ and $|d(v_1) - d(v_i)| \geq 2$, for every $v_i \in V(W_n), i \neq 1$, $|d(v_i) - d(v_j)| = 0, 2 \leq i, j \leq n,$ v_1 belongs to any of one the independent sets of the cycle. The number of independent sets in an even cycle is 2. Hence $\{\{v_1, v_2, v_4, \ldots, v_{n-1}\}, \{v_3, v_5, \ldots, v_n\}\}$ is an equitably independent partition of W_n. Thus $\pi_{ei}(W_n) = 2$ when n is odd.

The number of independent sets in an odd cycle is 3. Hence $\{\{v_1, v_2, v_4, \ldots, v_{n-2}\}, \{v_3, v_5, \ldots, v_{n-1}\}, \{v_n\}\}$ is an equitably independent partition of W_n. Thus $\pi_{ei}(W_n) = 3$ when n is even.

Theorem 2.15: If G is a unicyclic graph with cycle $C,$ then $1 \le \pi_{ei}(G) \le 3.$

Proof: By observation 2.4, $\pi_{ei}(G) \ge 1.$ Let $C = (v_1, v_2, \ldots v_n, v_1).$ Then $T_G = G - v_1 v_n$ is a tree of G. Let V_1 and V_2 be two independent sets of T. Clearly, v_1 belongs to any one of the sets V_1 and V_2, say V_1. If n is even, then $\{V_1, V_2\}$ is an equitably independent partition of G. On the other hand, if n is odd, $\{V_1 - \{v_n\}, V_2, \{v_n\}\}$ is an equitably independent partition of G. Thus $\pi_{ei}(G) \le 3.$Hence $1 \le \pi_{ei}(G) \le 3.$

3. Equitably independent partition of the Mycielskian Graph

For a graph $G = (V, E)$, the ***Mycielskian of*** G is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$ where $V' = \{v' \mid v \in V\}$ and is disjoint from V and edge set $E' = E \cup \{vw' \mid vw \in E\} \cup \{v'u \mid v' \in V'\}.$

Theorem 3.16: For any graph G**,** $\pi_{ei}(\mu(G)) \le \pi_{ei}(G) + 1.$

Proof:Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, $V(\mu(G)) = V \cup V' \cup \{u\}.$ Let $\{V_1, V_2, \ldots, V_k\}$ be an equitably independent partition of G. Since no vertex of V is adjacent to u, u belongs to any one of the set in the equitably independent partition. Further V' is an independent set. Hence $\{V_1 \cup \{u\}, V_2, \ldots V_k, V'\}$ is an equitably independent partition of $\mu(G)$. Thus $\pi_{ei}(\mu(G)) \le \pi_{ei}(G) + 1.$

Note: There exist graphs with $\pi_{ei}(\mu(G)) = \pi_{ei}(G) + 1.$ For the complete graph $K_3,\ \pi_{ei}(\mu(K_3)) = \pi_{ei}(K_3) + 1 = 3 + 1 = 4.$

Theorem 3.17: If G is a graph with $\delta(G) \ge 3$ and $\pi_{ei}(G) \ge 2$ then $\pi_{ei}(\mu(G)) \le \pi_{ei}(G).$

Proof: Let $\pi = \{V_1, V_2, \ldots, V_k\},\ k \ge 2$ be a minimum equitably independent partition of G. Since $\delta(G) \ge 3,$ the minimum degree of a vertex in V is 6 and the minimum degree of a vertex in V' is 3. Certainly, $(\pi - V_1 - V_2) \cup (V_1 \cup V') \cup (V_2 \cup \{u\})$ is an equitably independent partition of $\mu(G)$. Therefore $\pi_{ei}(\mu(G)) \leq \pi_{ei}(G)$.

Note: There exist graphs with $\pi_{ei}(\mu(G)) = \pi_{ei}(G)$. For the complete graph K_n, $n \geq 4$, $\pi_{ei}(\mu(G)) = \pi_{ei}(G) = n$.

Theorem 3.18: If G is a graph with $\delta(G) \geq 3$ and $\pi_{ei}(G) \geq 2$ and in which all the adjacent vertices are of distinct degree, then $\pi_{ei}(\mu(G)) \leq 2$.

Proof: Let $\pi = \{V_1, V_2, ..., V_k\}$, $k \geq 2$ be a minimum equitably independent partition of G. Since $\delta(G) \geq 3$, the minimum degree of a vertex in V is 6 and the minimum degree of a vertex in V' is 3. Since all the adjacent vertices of G are of distinct degree, any two adjacent vertices of V in $\mu(G)$ are inequitable. Hence $\{V_1 \cup V_2 \cup ... \cup V_k \cup \{u\}, V'\}$ is an equitably independent partition of $\mu(G)$. Hence $\pi_{ei}(\mu(G)) \leq 2$.

4. Realization Theorems

Theorem 4.19 : Given any two positive integers k and n such that $k \leq n$, there exists a graph G such that $\pi_{ei}(G) = k$ and $o(G) = n$.

Proof: **Case 1:** $k = 1, n \geq 1$.

Consider the totally disconnected graph G. Since for a totally disconnected graph $V(G)$ is an independent set, $\pi_{ei}(G) = 1$.

Case 2: $k = 2, n \geq 2$.

Consider the complete bipartite graph G with $\frac{n}{2}$ vertices in V_1 and $\frac{n}{2}$ vertices in V_2, when n is even and $\frac{n-1}{2}$ vertices in V_1 and $\frac{n+1}{2}$ vertices in V_2, when n is odd. Hence $|d(u) - d(v)| \leq 1$ for every $u \in V_1$ and $v \in V_2$. Thus $\pi_{ei}(G) = 2$.

Case 3: $k = n \geq 3$. Here the required graph G is the complete graph K_k.

Case 4: $n = k + 1$, $k \geq 3$. Here the required graph G is the complete graph K_k with one pendent edge attached with any one of the k vertex.

Case 5: $k \geq 3, n \geq 5$. The construction of the required graph G is as follows:

Consider the complete graph K_{k+1} with $k+1$ vertices. Let $V(K_k)=\{v_1,v_2,\ldots,v_{k+1}\}$. Introduce $n-k$ vertices namely $u_1,u_2,\ldots,u_{n-k}$. Now, join v_1 with $u_1,u_2,\ldots,u_{n-k}$. Hence $d(v_1)=k+n-k=n$ and $d(v_i)=k$ for all i, $2\le i\le k+1$. Thus $|d(v_1)-d(v_i)|=n-k\ge 2$ for all i, $2\le i\le k+1$.

Hence $\{\{v_1,v_2,u_1,u_2,\ldots,u_{n-k}\},\{v_3\},\ldots,\{v_{k+1}\}\}$ is a minimum equitably independent partition of G. Thus $\pi_{ei}(G)=k$ and $o(G)=n$.

5. Conclusion

Domination and coloring in graphs are the fastest growing area in graph theory and has been extensively studied in Chartrand and Zhang [2] and Haynes, Hedetniemi and Slater [4, 5]. In this article, we discussed the concept of equitably independent partition of a graph and its corresponding parameter namely equitably independent partition number. This paper can be further extended by analyzing the relation between equitably independent partition and equitable domination [6].

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