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Energy Restrictions and Renormalization on the Heptagon Star Fractal using Electrical Network Interpretation

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Abstract

The Delta-Wye Electrical transform is applied iteratively on the electrical network constructed on the Heptagon Star fractal to compute the resistance of the electrical network and the Energy renormalization factor by the concept of the electrical transform interpretation. There are several methods available to compute the resistance in any network, the Delta-Wye Electrical transform is the one of the methods. The Heptagon Star fractal is obtained by the method called the iterated function system which consists of seven similar contraction maps. The Delta-Wye transforms is applied iteratively on the Heptagon Star fractal, then the resistance and the energy renormalization factor are computed.

Keywords: Energy renormalization factor, Heptagon Star fractal, Delta-Wye transforms, Resistance.

I. Introduction

A Fractal system is a system characterized by a scaling law with non-integer exponent. Fractal systems are self-similar, which means a magnification of a small part is statistically equivalent to the whole. In the field of Mathematics, the research on analysis on fractals originated by the works of Kusuoka [10] and Goldstein [11]. The harmonic analysis on the fractal came from J.Kigami [2][7][9] and Robert S. Strichartz [1], they have defined the harmonic matrices[2] and the graph energy[1] on the Sierpinski gasket. Also they have established the Electrical network analysis on fractals. In any circuit we know the combining of the resistors in series and parallel for avoiding the complexity of any electrical circuit. Considering the two resistances R1 and R2, the combination or addition of the resistances in series is defined by $R=R_1+R_2$ and in parallel is $R=(R_1^{-1}+R_2^{-1})^{-1}$. When the electrical networks are more complicated by not in series or parallel connections, then the Δ-Y and Y-Δ transforms can be used to simplify the complexity of the network. The Δ-Y and Y-Δ transforms were first discovered by Kennelly in 1899[13]. When applying practically and repetitively, these transforms steadily reduce planar lattice into single bond, and thus it provides a powerful approach for the computation of the conductance of large networks. In this paper we consider the Heptagon Star for the analysis of the energy renormalization [14] using the self-similar fractal resistor networks (electrical network) called the Delta-Wye transform [5]. The networks are constructed with the property that each resistor in a network of a given order is a scalar multiple of the corresponding resistor in the next order network [3],[4]. The Heptagon is one of the easiest examples of a nested self-similar fractal [12]. The self similar resistance form is obtained on the Heptagon star fractal.

In this paper we will compute resistance and the energy renormalization factor on the self-similar fractal Heptagon star using the idea of electrical resistor networks. In section II, the graph energy is defined and the construction of the electrical Δ-Y transform is given. In Section III, the construction of the Heptagon star fractal is described. In Section IV, the Δ-Y transform is applied on the Heptagon Star fractal and the resistance and the energy renormalization factor of the network is computed as the main result.

II. Electrical Network Interpretation

A. Energy Restriction and Renormalization

The self similar energy constructed on a post critically finite fractal is given by

$$\varepsilon(u) = \sum_i r_i^{-1}\, \varepsilon\left(u \circ F_i\right) \qquad (2.1)$$

for certain resistance renormalization factors r_i which satisfies the condition that $0 < r_i < 1$. and ε is the limit of graph energies $E(u)$ defined on the pcf self similar fractal.

B. Graph Energy

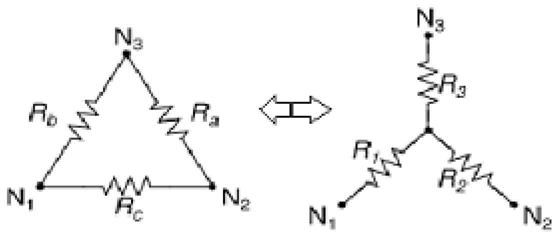
Consider a general finite, connected graph and the graph energy of the graph is defined by

$$E(u) = \sum_{x \sim y} c_{xy}\left(u(x) - u(y)\right)^2 \qquad (2.2)$$

Where c_{xy} is a positive function defined on the edges of the graph, which represents the conductance and its resistance of the electrical network is defined as the reciprocal of the conductance. Consider the each graph as a electrical network and each of its vertices as nodes and the edges as resistors linking the nodes with the set resistances. And *u(x)* and *u(y)* are the voltages or energy functions given to the nodes *x* and *y* respectively. The energy $c_{xy}\left(u(x) - u(y)\right)^2$ will be produced from each of the resistors by the current flow $\left(\dfrac{u(x) - u(y)}{R_{xy}}\right)$.

C. Electrical Transform - Δ-Y transform

This transform is a mathematical method to simplify the analysis of any electrical network. The name is obtained from the shape of the circuit diagrams, which seem to be the letter Y and the Greek capital letter Δ respectively. Totally three resistors are used in both Y and Δ circuits. Each resistor gives resistance in opposition to the current flow through them and it is symbolized by the letter 'R' [6]. The basic concept of the Δ-Y transform is, the triangular arrangement of resistors, say R_a, R_b and R_c can be replaced by a star within the circuit, with resistances R_1,R_2 and R_3, such that each resistance between any of the two vertices among the three in the triangle and the star are the identical [8].



The Δ-Y Transform

The system equations are used to transform from Δ-load to Y-load circuit is given as equation.2.3,

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}$$
$$R_2 = \frac{R_a R_c}{R_a + R_b + R_c}$$
$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c} \qquad (2.3)$$

The general formula for the computation of resistances is,

$$R_\Delta = \frac{R_P}{R_{opposite}} \quad Where\ R_P = R_1R_2 + R_2R_3 + R_3R_1$$

R_P is the sum of the products of every pair of the resistances in the Y circuit and $R_{opposite}$ is the amount of resistance of the vertex in the Y circuit which is opposite the edge with R_Δ .

The system of equations to transform Y-load to Δ-load circuit is given as equation 2.4.,

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$
$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$
$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} \qquad (2.4)$$

III. Construction of the Heptagon Star Fractal

The Heptagon star or Heptagram is a seven point star drawn with seven straight strokes. The Heptagon star is a fractal F with self-similarity which is a compact metric space made of seven scaled copies given as $F_1, F_2, \ldots, F_7$ of itself, that is, $F = \bigcup_{j=1}^{7} F_j$

The scaling factor of Heptagon star is 2*x*+2, where *x* is the sine of the rotation angle. The angle is half the interior angle, which is 180^0 minus the exterior angle, which in turn is 720^0/n. Hence the formula is s=2+2(sin(90^0-360^0/n)), which is simplified to s=2+2(cos(360^0/n)). The Heptagon Star is obtained seven homotheties with the contraction ratio 1/3.4698 and with the rotation angle 60^o. The Heptagon Star is obtained by the IFS $F_i =$

$\frac{1}{3.4698}(x - p_i) + p_i, i = 1\ to\ 7$ where p_i's are the vertices of the regular Heptagon [1]. The graph of G_0 and G_1 are given in Figure 1.The graph G_0 is the basic triangle which contains three vertices as $G_0=\{X_1,X_2,X_3\}$ and the graph G_1 is the Heptagon Star with the vertex set $G1=\{X_1,X_2,X_3,X_4,X_5,X_6,X_7\}$.

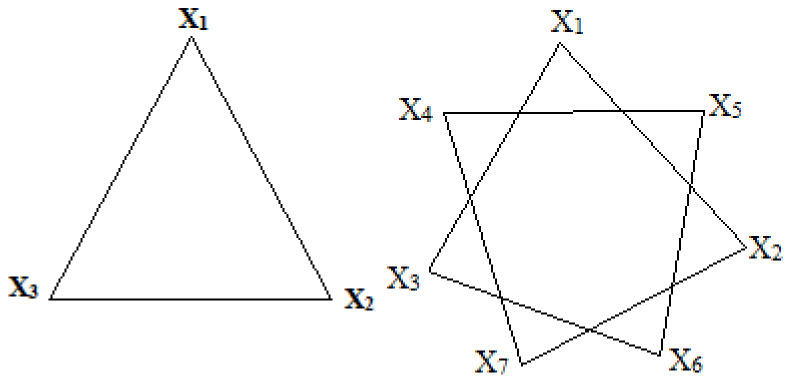


Figure 3.1.The Graph of G_0 and G_1 of Heptagon

The Heptagon Star with $N_0=3$ have a single reflection symmetry. We could look at a more general situation with two choices of r_i, 1 unit for the cell containing boundary points and ρ for the other cells.

IV. Electrical Δ-Y transform on the Heptagon Star

The Δ-Y transform is applied in the following Heptagon Star Figure 4.1(a), the equivalent network with three vertices is obtained. We consider the unit resistance in the outer sides of the Heptagon Star and ρ is the resistance in the inner sides of the Heptagon Star. Applying the Δ-Y transform in Figure 4.1(b), the resultant resistances is given in Figure 4.1 (c).

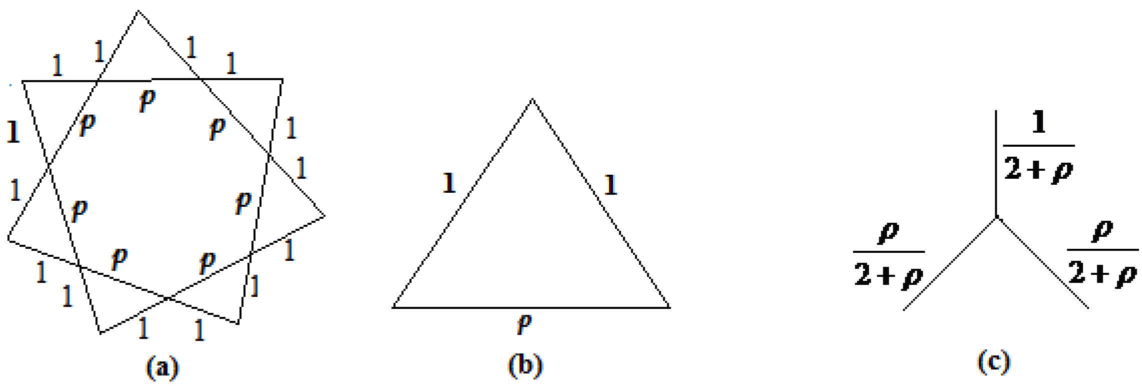


Figure 4.1

Applying the Δ-Y transform in Figure 4.1(a), the seven Y shapes are formed from the seven Delta shapes in the corners of the Heptagon star. As given in Figure 4.2(b), by pruning and adding resistances in series in Figure 4.2(a), we obtain Figure 4.2(c). Using another Δ-Y transform yields the Figure 4.2(d), and adding resistances in series yields Figure 4.2(e).

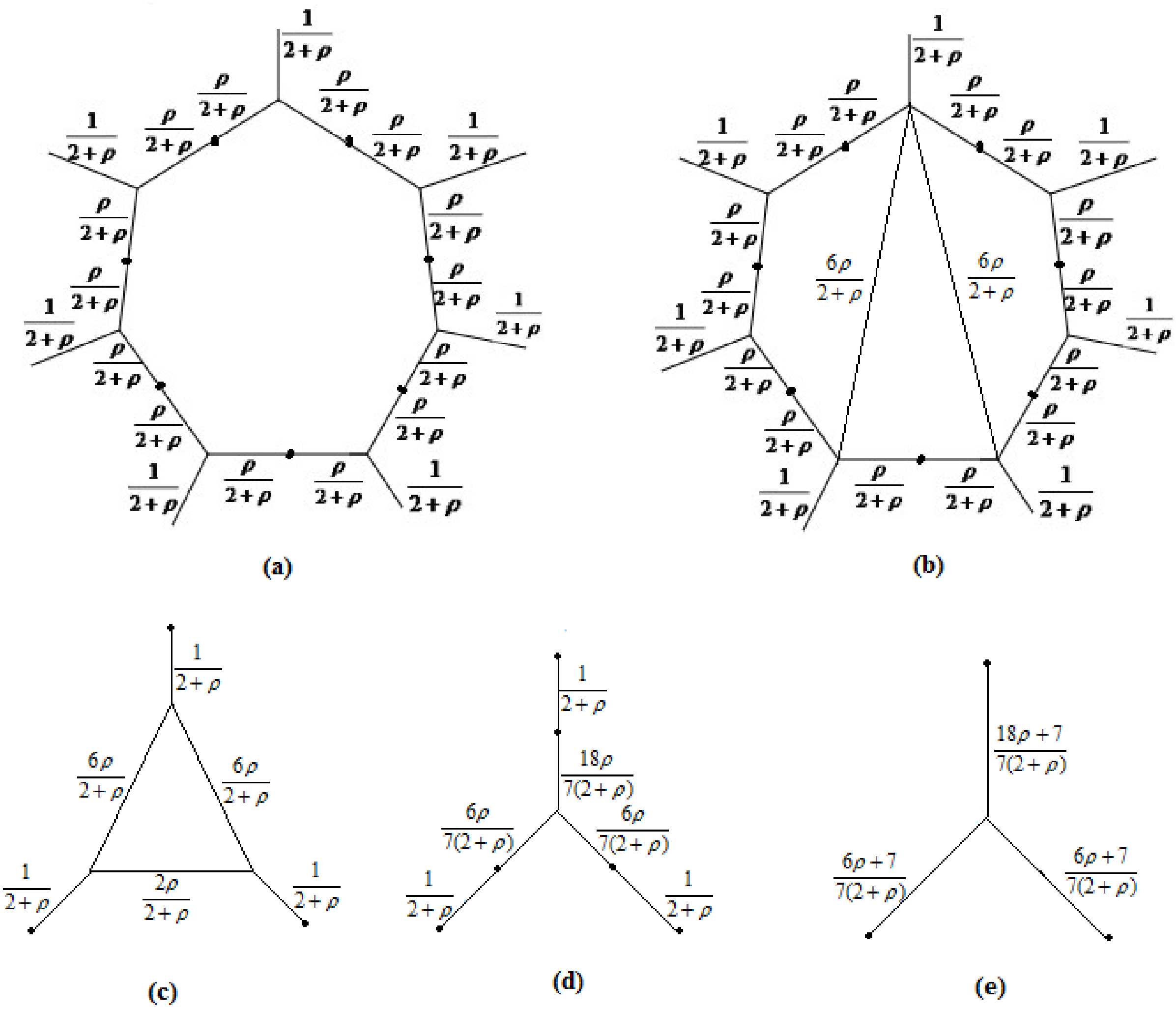


Figure 4.2

The renormalization equation says that this last network given in Figure 4.2(e) must be a multiple of the network in Figure 4.1(c), which yields the equation,

$$\frac{6\rho+7}{18\rho+7}=\rho \qquad \Rightarrow \qquad 18\rho^2+\rho-7=0$$

Solving this equation, we obtain the unique positive solution as, $\rho=\dfrac{\sqrt{505}-1}{36}=0.59645$.

The conductance defined on the network is

$$c_{xy} = \frac{1}{\rho} = \frac{36}{\sqrt{505}-1}\ \frac{\sqrt{505}+1}{\sqrt{505}+1} = \frac{\sqrt{505}+1}{14} = 1.67659$$

And the resistance is, $r_i = \dfrac{7}{18\rho+7} = \dfrac{7}{18\left(\dfrac{\sqrt{505}-1}{36}\right)+7} = \dfrac{\sqrt{505}-13}{24} \approx 0.39468.$

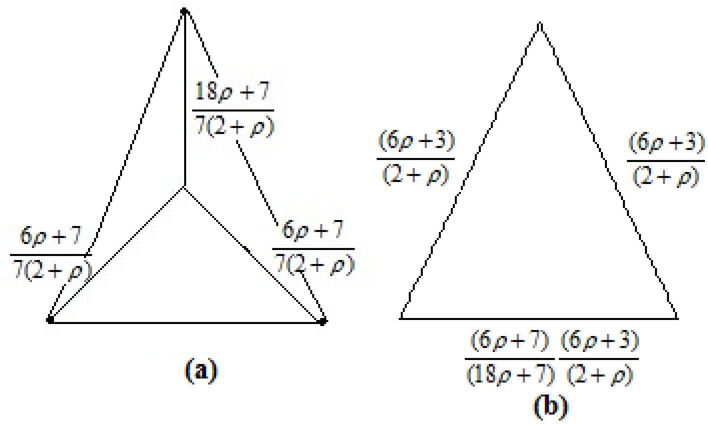


Figure 4.3

Again we apply the inverse of Δ-Y transform in Figure 4.2(e) to obtain Figure 4.3(a) as the network on Figure 4.1(b) and the track of resistances are kept along the way; the values are multiplied by $\dfrac{(6\rho+3)}{(2+\rho)} = 2.5337$, which is the renormalization factor in this transformation on the regular self similar Heptagon fractal.

V. Conclusion

The electrical network Δ-Y transform can be applied iteratively on the electrical network constructed on the self-similar Heptagon Star fractal and the resistance of the network is computed. The conductance of the electrical network is then calculated as the reciprocal of ρ. Finally in this paper the Energy renormalization factor of the electrical network is calculated by the concept of the electrical transform interpretation.

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