On generalized Differential and Integral operators
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Abstract: In the present paper we define a generalized Differential operator $D_{\alpha,\lambda, l}^k$ and Integral operator $I_{\alpha,\lambda, l}^k$ using convolution. The Differential operator generalizes various operators studied earlier by Al–Oboudi [1], Catas [3], Cho and Kim [4], Cho and Srivastava [5], Maslina Darus and Rabha Ibrahim [6], Salagean [8], Uralegaddi and Somanatha [12]. Further using these operators we define new subclasses of analytic functions [13] and we obtain sufficient condition for functions belonging to these classes. Several interesting consequences of our results are also pointed out.

Keywords: Univalent functions, Convolution, Generalized Differential and Integral operators.

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1. INTRODUCTION

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$  \hspace{1cm} (1.1)

which are analytic and univalent in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$.

Definition 1.1: Given two functions $f$ and $g$ in the class $A$, where $f$ is defined by (1.1) and $g$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

the convolution or Hadamard product $f \ast g$ is defined by the power series

$$(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \hspace{1cm} (z \in U).$$

Now we define the generalized differential operator [9] $D_{\alpha,\lambda, l}^k : A \rightarrow A$ such that

$$D_{\alpha,\lambda, l}^k = (\Psi(z) \ast \Psi(z) \ast \ldots \ast \Psi(z)) \ast f(z)$$

$k$ times

where
\[ \psi(z) = \frac{1}{l+1} \left[ \frac{z(\lambda - \alpha + 1)}{(1-z)^2} - \frac{z(\lambda - \alpha + l)}{(1-z)} \right] \]

and

\[ D^k_{a,\lambda,l}f(z) = z + \sum_{n=2}^{\infty} \left[ \frac{(n-1)(\lambda - \alpha) + l + n}{l+1} \right]^k a_n z^n \] (1.2)

Where \( \alpha \geq 0, \lambda \geq 0, n \in \mathbb{N}_0, \ l \geq 0. \)

**Remark:** By giving specific values to \( \alpha, \lambda, l \) and \( k \) we obtain the various operators studied earlier by Al–Oboudi [1], Catas [3], Cho and Kim [4], Cho and Srivastava [5], Maslina Darus and Rabha Ibrahim [6], Salagean [8], Uralegaddi and Somanatha [12].

Now we introduce a new subclass \( M^k_{a,\lambda,l}(\mu) \) using the generalized differential operator \( D^k_{a,\lambda,l}f(z) \) as follows

**Definition 1.2:** A function \( f(z) \in A \) is said to be in the class \( M^k_{a,\lambda,l}(\mu) \) if it satisfy the inequality

\[ \Re \left\{ \frac{z(D^k_{a,\lambda,l}f(z))'}{D^k_{a,\lambda,l}f(z)} \right\} < \mu, \quad (z \in U) \]

for some \( \mu (\mu > 1) \).

**Definition 1.3:** A function \( f(z) \in A \) is said to be in the class \( N^k_{a,\lambda,l}(\mu) \) if it satisfy the inequality

\[ \Re \left\{ \frac{z(D^k_{a,\lambda,l}f(z))''}{(D^k_{a,\lambda,l}f(z))'} \right\} < \mu, \quad (z \in U) \]

for some \( \mu (\mu > 1) \).

we have \( f \in N^k_{a,\lambda,l}(\mu) \) if and only if \( z f' \in M^k_{a,\lambda,l}(\mu) \).

**Remark:** The above classes reduces to the subclasses studied by

1) Uralegaddi et al. [10,11], for \( 1 < \mu \leq \frac{4}{3} \) and \( k=0 \)
2) Owa and Nishiwaki [7], for \( \mu > 1 \) and \( k=0 \)
3) Bulut [2], for \( \mu > 1, \alpha=1 \) and \( l=0 \)
4) Maslina Darus and Rabha Ibrahim [6], for \( l=0 \).
2. MAIN RESULTS

In this section using coefficient inequalities we obtain the sufficient conditions for a function \( f(z) \) to be in the classes \( M^{k}_{\alpha,\lambda,l}(\mu) \) and \( N^{k}_{\alpha,\lambda,l}(\mu) \).

**Theorem 2.1:** If the function \( f(z) \) belonging to A satisfies the inequality

\[
\sum_{n=2}^{\infty} \left| \left( \frac{(n-1)(\lambda - \alpha) + l + n}{l + 1} \right)^k \right| \{ (n - k) + |n + k - 2\mu| \} |a_n| \leq 2(\mu - 1) \tag{2.1}
\]

for some \( 0 \leq k \leq 1 \) and \( \mu > 1 \), then \( f(z) \in M^{k}_{\alpha,\lambda,l}(\mu) \).

**Proof:** Suppose the inequality (2.1) holds, then it sufficient to show that

\[
\left| \frac{z(D_{\alpha,\lambda,l}^k f(z))'}{D_{\alpha,\lambda,l}^k f(z)} - k \right| < 1, \quad (z \in U)
\]

Note that

\[
1 - k + \sum_{n=2}^{\infty} (n - k) \left| \left( \frac{(n-1)(\lambda - \alpha) + l + n}{l + 1} \right)^k \right| a_n z^{n-1} \\
\leq \frac{1 - k + \sum_{n=2}^{\infty} (n - k) \left| \left( \frac{(n-1)(\lambda - \alpha) + l + n}{l + 1} \right)^k \right| a_n |z|^{n-1}}{2\mu - 1 - k - \sum_{n=2}^{\infty} |n + k - 2\mu| \left| \left( \frac{(n-1)(\lambda - \alpha) + l + n}{l + 1} \right)^k \right| a_n |z|^{n-1}}
\]
now the last expression is bounded above by 1 if
\[
1 - k + \sum_{n=2}^{\infty} (n-k) \left| \left( \frac{(n-1)(\lambda - \alpha) + l + n}{l + 1} \right)^k \right| a_n
\]

which is equivalent to (2.1). This completes the proof.

**Theorem 2.2:** If the function \( f(z) \) belonging to A satisfies the inequality
\[
\sum_{n=2}^{\infty} \left| \left( \frac{(n-1)(\lambda - \alpha) + l + n}{l + 1} \right)^k \right| a_n \leq 2(\mu - 1)
\]
for some \( 0 \leq k \leq 1 \) and \( \mu > 1 \), then \( f(z) \in N_{\alpha,\lambda,l}^k(\mu) \).

Remark: Upon setting suitable values for the parameters in Thm(2.1) and Thm(2.2), we get the results obtained in [6] and [7].

### 3. INTEGRAL OPERATOR

In this section analogous to the differential operator \( D_{\alpha,\lambda,l}^k \), we define a new Integral \( I_{\alpha,\lambda,l}^k : A \to A \) as follows:

Let
\[
F(z) = \left( \psi(z) \cdot \psi(z) \cdots \psi(z) \right) = z + \sum_{n=2}^{\infty} \left[ \frac{(n-1)(\lambda - \alpha) + l + n}{l + 1} \right]^k z^n
\]
where
\[
\psi(z) = \frac{1}{l + 1} \left[ \frac{z(\lambda - \alpha + 1)}{(1-z)^2} - \frac{z(\lambda - \alpha + l)}{(1-z)} \right]
\]
Now we define the Integral operator $I_{\alpha,\lambda,l}^k$ such that

$$I_{\alpha,\lambda,l}^k = \frac{F(z)}{z} \ast f(z), \quad (z \in U) \quad (3.1)$$

$$I_{\alpha,\lambda,l}^k f(z) = z + \sum_{n=2}^{\infty} \left[ \frac{l + 1}{(n-1)(\lambda - \alpha) + l + n} \right] a_n z^n \quad (3.2)$$

Where $\alpha \geq 0$, $\lambda \geq 0$, $n \in \mathbb{N}$, $l \geq 0$.

**Remark:** For $l = 0$ this operator reduces to the integral operator studied in [6] and for $\alpha = \lambda$ this reduces to the integral operator studied in [8].

**Lemma 3.1:** Let $f \in A$. Then

a) $I_{\alpha,\lambda,l}^0 f(z) = f(z)$

b) $I_{\alpha,\alpha,0}^1 f(z) = \int_0^z \frac{f(t)}{t} dt$

**Proof:** Let $f \in A$. Then

a) $I_{\alpha,\lambda,l}^0 f(z) = z + \sum_{n=2}^{\infty} a_n z^n = f(z)$

b) $\int_0^z \frac{f(t)}{t} dt = \int_0^z \left[ 1 + \sum_{n=2}^{\infty} a_n t^{n-1} \right] dt = z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n = I_{\alpha,\alpha,0}^1 f(z)$.

Using the generalized Integral operator we define the new subclasses as follows.

**Definition 3.2:** A function $f(z) \in A$ is said to be in the class $L_{\alpha,\lambda,l}^k(\mu)$ if it satisfy the inequality

$$\Re \left\{ \frac{z(I_{\alpha,\lambda,l}^k f(z))'}{(I_{\alpha,\lambda,l}^k f(z))'} \right\} < \mu, \quad (z \in U)$$

for some $\mu (\mu > 1)$.

**Definition 3.3:** A function $f(z) \in A$ is said to be in the class $S_{\alpha,\lambda,l}^k(\mu)$ if it satisfy the inequality

$$\Re \left\{ \frac{z((I_{\alpha,\lambda,l}^k f(z))''}{(I_{\alpha,\lambda,l}^k f(z))'} \right\} < \mu, \quad (z \in U)$$

for some $\mu (\mu > 1)$. 

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We have \( f \in S^k_{\alpha,\lambda,l}(\mu) \) if and only if \( z f' \in L^k_{\alpha,\lambda,l}(\mu) \).

**Remark:** For \( l = 0 \) and \( \alpha = \lambda \) the above classes reduces to the subclasses studied in [6] and [8] respectively.

**Theorem 3.4:** If the function \( f(z) \) belonging to A satisfies the inequality

\[
\sum_{n=2}^{\infty} \left| \frac{l+1}{(n-1)(\lambda - \alpha) + l + n} \right|^k \{(n - k) + |n + k - 2\mu|} a_n \leq 2(\mu - 1)
\]

for some \( 0 \leq k \leq 1 \) and \( \mu > 1 \), then \( f(z) \in L^k_{\alpha,\lambda,l}(\mu) \).

**Theorem 3.5:** If the function \( f(z) \) belonging to A satisfies the inequality

\[
\sum_{n=2}^{\infty} \left| \frac{l+1}{(n-1)(\lambda - \alpha) + l + n} \right|^k n\{(n - k) + |n + k - 2\mu|} a_n \leq 2(\mu - 1)
\]

for some \( 0 \leq k \leq 1 \) and \( \mu > 1 \), then \( f(z) \in S^k_{\alpha,\lambda,l}(\mu) \).

**REFERENCES:**


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