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Numerical solution for time fractional gas dynamics equations through reduced differential transform coupled with fractional complex transform

T.R.Ramesh Rao
Department of Mathematics & Actuarial Science
B.S.Abdur Rahman Crescent University
Chennai-600 048.

Abstract: An analytical method based on reduced differential transform method coupled with fractional complex transform is described. The present technique has uniqueness in converting the fractional differential equation into ordinary differential equation in straight forward method without any discretization, linearization and perturbation of the problem. The fractional derivatives are described in modified Riemann-Liouvilles sense. The proposed method is an alternative approach which reduces the size of the computational work for finding the approximate analytical solution of time fractional nonlinear homogeneous and nonhomogeneous gas dynamics equations in effective manner. Finally some plots are presented to discuss the variation of fractional order α through symbolic computation software Mathematica 7.

Keywords: Reduced differential transform, fractional derivatives, complex fractional transform, Gas dynamics equation

1. Introduction

In this paper, we consider the time fractional gas dynamics equation of the following type:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2}(\mathrm{u}^2)_\mathrm{x} - \mathrm{u}(1-\mathrm{u}) = \mathrm{g}(\mathrm{x},\mathrm{t}), \qquad \mathrm{t} > 0,\ 0 < \alpha \leq 1 \tag{1}$$

subject to the initial condition $u(x,0) = f(x)$, where α is a parameter describing the order of the fractional derivative and g(x,t) is a known function. When $\alpha = 1$, the Eqn. (1) reduces to the gas dynamics equation of standard type.

Gas dynamics equations are the mathematical expressions based on the physical laws of conservation of mass, conservation of momentum, conservation of energy etc. The few types of gas dynamics equations in physics have been solved by Elizarova [13], Jafari et al [15], Evans and Bulut [19], Polyanin and Zaitsev [20] by applying different analytical and approximation methods. Recently several analytical / numerical methods have been developed for the solution of gas dynamic equations. S.Das et al [2] have used differential transform method to obtain the semi analytical solution of nonlinear homogeneous time – fractional gas dynamics equations. Many researchers have paid attention in studying the coupling technique for handling the fractional differential equations i.e. combination of two analytical methods [4, 12, 17, 18] to solve nonlinear gas dynamics equations. Mohammed Tamsitr et al [1] used a recent semi analytical method referred as fractional reduced differential transform method to obtain the approximate analytical solution of time fractional gas dynamics equations. A new iterative method for fractional gas dynamics equation[21] was proposed by Mohammed. S.Al-luhaibi [3].

2. Basic idea of reduced differential transform

The Reduced Differential Transform method (RDTM) was first proposed by Keskin [9, 10, 11] for solving nonlinear partial differential equations. This method based on Taylor series gives approximate analytical solution in the form of convergent series. Later many researchers have improved and modified this method to obtain rapid convergent series solution of nonlinear problems. The brief analysis of RDTM is as follows:

Consider a function u(x, t) of two variables and assume that it can be represented as a product of two single variable functions, defined as u(x, t) = f(x) g (t). On the basis of the properties the one dimensional differential transform, the function u(x, t) can be represented as

$$u(x,t) = \sum_{h=0}^{\infty} F(h)x^h \sum_{k=0}^{\infty} G(k)t^k = \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} U(h,k)x^h t^k \quad (2)$$

Where$U(h,k) = F(h)G(k)$ is called the spectrum of $u(x,t)$.

The basic definitions and properties of reduced differential transform (RDTM) are introduced below.

The reduced differential transform of u(x, t) at t = 0 is defined as

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{t=0} \quad (3)$$

Where u(x, t) is the given function and $U_k(x)$ is the transformed function.

The reduced differential inverse transform of $U_k(x)$is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k \quad (4)$$

and from (1) and (2), we have

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{t=0} t^k \quad (5)$$

Theorem 1. If $w(x,t) = u(x,t) + v(x,t)$ then $W_k(x) = U_k(x) + V_k(x)$

Theorem 2. If $w(x,t) = \alpha u(x,t)$ then$W_k(x) = \alpha U_k(x)$

Theorem 3. If $\mathrm{w}(\mathrm{x},\mathrm{t}) = \alpha \frac{\partial^n u(x,t)}{\partial t^n}$ then $\mathrm{W_k}(\mathrm{x}) = \alpha \frac{(k+n)!}{k!} \mathrm{U_{k+n}}(\mathrm{x})$

Theorem 4. If $\mathrm{w}(\mathrm{x},\mathrm{t}) = x^m t^n$ then $\mathrm{W_k}(\mathrm{x}) = x^m \delta(k-n) = \begin{cases} x^m & if\ k = n \\ 0 & if\ k \neq n \end{cases}$

Theorem 5. If $\mathrm{w}(\mathrm{x},\mathrm{t}) = \alpha \frac{\partial^n u(x,t)}{\partial x^n}$ then $\mathrm{W_k}(\mathrm{x}) = \alpha \frac{\partial^n U_k(x)}{\partial x^n}$

Theorem 6. If $\mathrm{w}(\mathrm{x},\mathrm{t}) = \mathrm{u}(x,t)v(x,t)$ then $\mathrm{W_k}(\mathrm{x}) = \sum_{k_1=0}^{k} U_{k_1}(x)V_{k-k_1}(x)$

Theorem 7. If $\mathrm{w}(\mathrm{x},\mathrm{t}) = t^n \mathrm{u}(x,t)$ then $\mathrm{W_k}(\mathrm{x}) = U_{k-n}(x)$

Theorem 8. If $\mathrm{w}(\mathrm{x},\mathrm{t}) = x^m t^n \mathrm{u}(x,t)$ then $\mathrm{W_k}(\mathrm{x}) = x^m U_{k-n}(x)$

3. Brief analysis of complex fractional transform

3.1 Jumaries fractional derivatives

Recently many researchers have paid much attention in studying the numerical methods for finding the solution of fractional differential equations through complex transform method. The fractional complex transform is the simplest approach in fractional calculus which converts the fractional differential equation into integer order differential equation making the solution procedure extremely simple [5,6,7,8]. Moreover the fractional derivatives are defined in Jumaries Modified Reiemann-Louvilles sense which is relatively a simple approach and is easier to handle the problems.

Jumaries fractional derivative is a modified Riemann-Liouvilles derivative of order $'\alpha'$ defined as

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_0^t (t-\zeta)^{-\alpha-1}[f(\zeta)-f(0)]d\zeta & \alpha < 0 \\ \frac{1}{\Gamma(-\alpha)}\frac{d}{dt}\int_0^t (t-\zeta)^{-\alpha-1}[f(\zeta)-f(0)]d\zeta, & 0<\alpha<1 \\ \left[f^{(\alpha-m)}(t)\right]^{(m)}, & m \le \alpha \le m+1, \quad m \ge 1 \end{cases} \tag{6}$$

where $f: R \to R$ is a continuous function.
We list some important properties for the modified Riemann-Liouvilles derivatives as follows:

$(i)\ D_t^\alpha(c) = 0, \alpha > 0, c$ is a constant (7)

$(ii)\ D_t^\alpha[cf(t)] = cD_t^\alpha f(t), \alpha > 0$ (8)

$(iii)\ D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha},\ \beta > \alpha > 0$ (9)

$(ii)\ D_t^\alpha[f(t)g(t)] = [D_t^\alpha f(t)]g(t) + f(t)[D_t^\alpha g(t)]$ (10)

$(v)\ D_t^\alpha\left[f\left(h(t)\right)\right] = f_h'\left(h(t)\right)D_t^\alpha h(t)$ (11)

3.2 Fractional complex transform method

The fractional complex transform was first proposed by Zeng Biao Li et al [14] and is defined as

$$T = \frac{at^\alpha}{\Gamma 1+\alpha} \tag{12}$$

$$X = \frac{bx^\beta}{\Gamma 1+\beta} \tag{13}$$

$$Y = \frac{cy^\gamma}{\Gamma 1+\gamma} \tag{14}$$

$$Z = \frac{dz^\delta}{\Gamma 1+\delta} \tag{15}$$

where a, b, c, d are unknown constants and $0 < \alpha \le 1, 0 < \beta \le 1, 0 < \gamma \le 1$ and $0 < \delta \le 1$.

4. Numerical Applications

In this section, we demonstrate the effectiveness of the proposed method by presenting three types of fractional gas dynamics equations.

Example 4.1 Consider the following fractional homogeneous nonlinear gas dynamics equation [1, 2, 3, 4, 16, 17].

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2}(u^2)_x - u(1-u) = 0 \qquad t > 0,\ 0 < \alpha \le 1 \tag{16}$$

with the initial condition

$$u(x,0) = e^{-x} \tag{17}$$

Applying the transformation [14], we get the following integer order PDE:

$$\frac{\partial u}{\partial T} + \frac{1}{2}(u^2)_x - u(1-u) = 0 \tag{18}$$

By taking the reduced differential transform on both sides of Eqns. (18) and (17), we obtain the following recurrence relation:

$$(k+1)U_{k+1}(x) = U_k(x) - \sum_{k_1=0}^k U_{k_1}(x)U_{k-k_1}(x) - \sum_{k_1=0}^k U_{k_1}(x)\frac{\partial}{\partial x}U_{k-k_1}(x) \tag{19}$$

$$U_0(x) = e^{-x} \tag{20}$$

Where $U_k(x)$ is the transformed value of $u(x,T)$

Now, substituting the Eqn. (20) in to the Eqn. (19) and by straight ward iterative procedure, yields

$$U_1(x) = e^{-x},\ \ U_2(x) = \frac{e^{-x}}{2},\ U_3(x) = \frac{e^{-x}}{6},\ \ U_4(x) = \frac{e^{-x}}{24}, ... \tag{21}$$

From (4) and (12), we have

$$u(x,t) = e^{-x} + e^{-x}\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right) + \frac{e^{-x}}{2!}\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right)^2 + \frac{e^{-x}}{3!}\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right)^3 + \frac{e^{-x}}{4!}\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right)^4 + \cdots \tag{22}$$

and this series is convergent to the exact solution

$$u(x,t) = e^{-x+\frac{t^\alpha}{\Gamma 1+\alpha}}, \quad t > 0,\ 0 < \alpha \le 1, \tag{23}$$

This result shows excellent agreement with one obtained by FRDTM[1], DTM[2], NIM[3], LTNHPM[4], FHATM[16], HPSTM[17] when $\alpha = 1$. Fig 1 (a) shows the plot of the exact solution $u(x,t)$ verses t for $\alpha = 0.25, 0.5, 0.75, 1$.

Example 4.2 Consider the following homogeneous nonlinear time fractional gas dynamics equation [1, 16, 18]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2}(u^2)_x - u(1-u)\log a = 0, \qquad t > 0,\ 0 < \alpha \le 1 \tag{24}$$

With the initial condition

$$u(x,0) = a^{-x}\ , a > 0 \tag{25}$$

Applying the transformation [19], we obtain the following integer order PDE:

$$\frac{\partial u}{\partial T} + \frac{1}{2}(u^2)_x - u(1-u)\log a = 0, \qquad t > 0,\ 0 < \alpha \le 1 \tag{26}$$

Taking known reduced differential transform on both sides of the Eqns. (26) and (25), we get

$$U_{k+1}(x) = \frac{1}{(k+1)}\left[U_k(x)loga - \sum_{k_1=0}^{k} U_{k_1}(x)U_{k-k_1}(x)loga - \sum_{k_1=0}^{k} U_{k_1}(x)\frac{\partial}{\partial x}U_{k-k_1}(x)\right] \tag{27}$$

$$U_0(x) = a^{-x} \tag{28}$$

Substituting (28) in to (27) and using the recurrence relation, we get

$$U_1(x) = a^{-x}loga\ ,\ \ U_2(x) = \frac{a^{-x}(loga)^2}{2!}\ ,\ \ U_3(x) = \frac{a^{-x}(loga)^3}{3!},\ \ U_4(x) = \frac{a^{-x}(loga)^3}{3!}, \ldots.$$

From (4) and (12), we have

$$u(x,t) = a^{-x} + a^{-x}(loga)\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right) + \frac{a^{-x}}{2!}(loga)^2\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right)^2 + \frac{a^{-x}}{3!}(loga)^3\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right)^3 + \frac{a^{-x}}{4!}(loga)^4\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right)^4 + \cdots \tag{29}$$

Therefore we obtain the exact solution

$$u(x,t) = a^{-x+\frac{t^\alpha}{\Gamma 1+\alpha}}, \qquad t > 0,\ 0 < \alpha \le 1 \tag{30}$$

Example 4.3 Consider the following nonhomogeneous nonlinear time fractional gas dynamics equation [2, 4]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2}(u^2)_x + u(1-u) = -e^{-x+t} \qquad t > 0,\ 0 < \alpha \le 1 \tag{31}$$

with the initial condition

$$u(x,0) = 1 - e^{-x} \tag{32}$$

Applying the transformation [14], we get the following integer order PDE:

$$\frac{\partial u}{\partial T} + \frac{1}{2}(u^2)_x + u(1-u) = -e^{-x+t} \tag{33}$$

By taking the reduced differential transform method to both sides of the Eqns. (33) and (32), we obtain the following recurrence relation:

$$(k+1)U_{k+1}(x) = U_k(x) - \sum_{k_1=0}^{k} U_{k_1}(x)U_{k-k_1}(x) - \sum_{k_1=0}^{k} U_{k_1}(x)\frac{\partial}{\partial x}U_{k-k_1}(x) - \frac{e^{-x}}{k!} \tag{34}$$

$$U_0(x) = 1 - e^{-x} \tag{35}$$

By iterative calculations on (34) and (35), we have

$U_1(x) = -e^{-x}\ ,\ U_2(x) = -\frac{e^{-x}}{2},\ U_3(x) = -\frac{e^{-x}}{6},\ U_4(x) = -\frac{e^{-x}}{24}$, ... and so on.

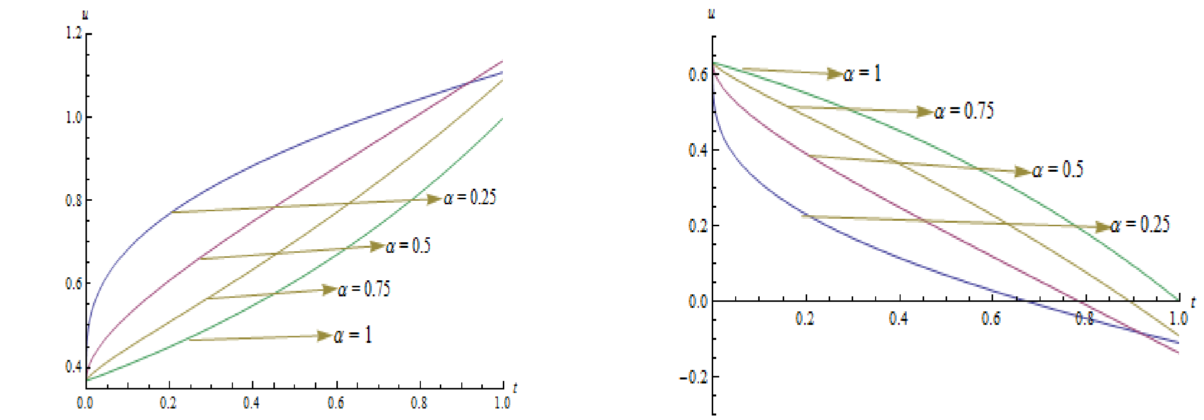
Substituting the above iterative values into (4) and using (12), we have

$$u(x,t) = 1 - e^{-x} - e^{-x}\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right) - \frac{e^{-x}}{2!}\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right)^2 - \frac{e^{-x}}{3!}\left(\frac{t^\alpha}{\Gamma 1+\alpha}\right)^3 - \cdots \tag{36}$$

and we obtain the closed form solution

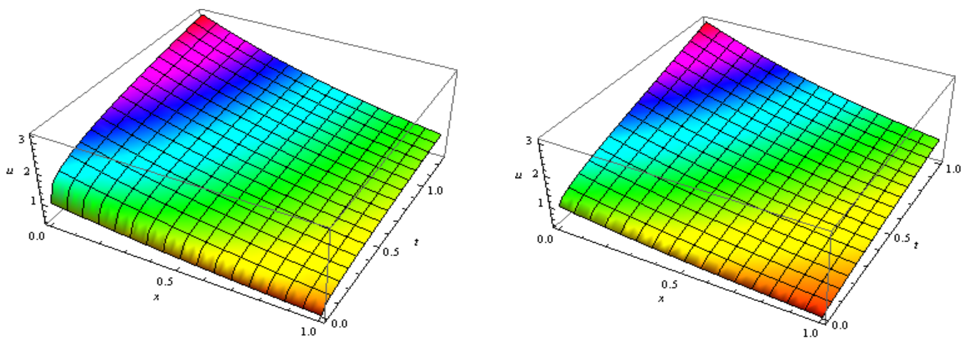
$$u(x,t) = 1 - e^{-x} + \frac{t^{\alpha}}{\Gamma 1+\alpha} \qquad t > 0,\ 0 < \alpha \leq 1 \tag{37}$$

The same solution was obtained by using LTNHPM [4] with $\alpha = 1$ and decomposition method [19] when $\alpha = 1$. Fig 1 (b) shows the plot of the exact solution $u(x,t)$ verses t for $\alpha = 0.25, 0.5, 0.75, 1$. Moreover, Fig 2(a) – 2(d) and Fig 3(a) –3 (d) explores the exact solution of the problems 4.1 and 4.3 for α = 0.25, 0.5, 0.75 and 1.We observed when the fractional order α is closed to 0, the solution curve is very smooth than α is nearer to 1. This will help us for better understanding of time fractional derivatives and its relation to the mathematical model arise from real life problems.

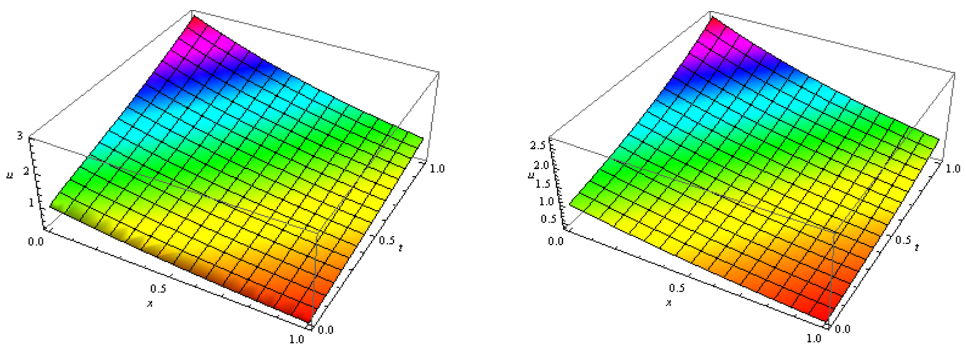


(a) **(b)**

Fig. 1. Plot of u(x,t) vs t at different values of α for example (a) 4.1 (b) 4.3

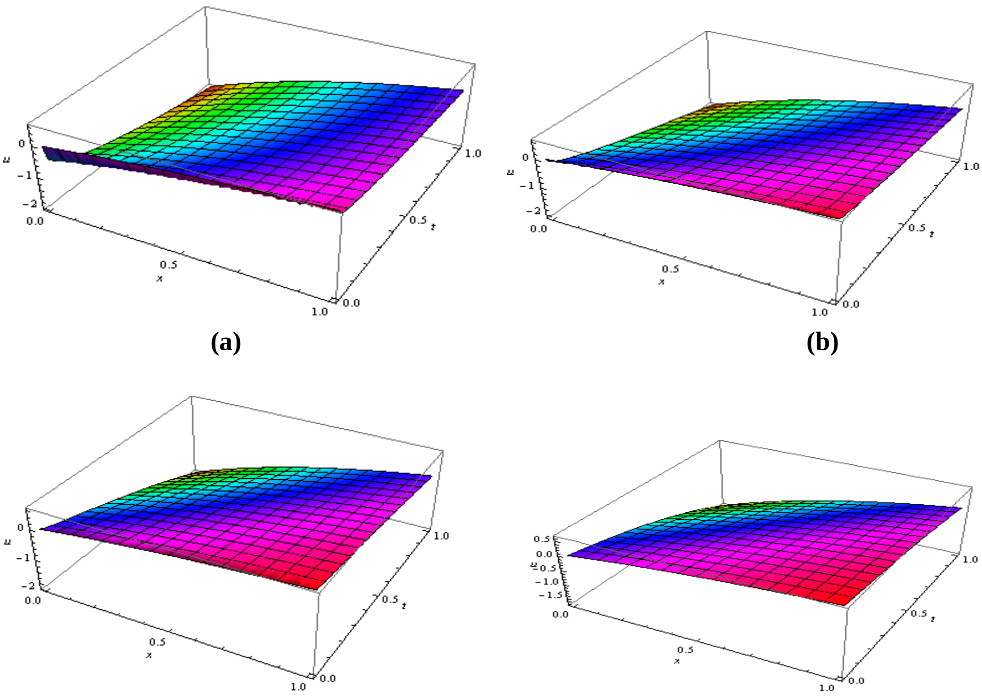


(a) **(b)**



(c) **(d)**

Fig. 2. Plot of u (x, t) w.r.t x and t for example 4.1 at (a) α = 0.25 (b) α = 0.5 (c) α = 0.75 (d) α = 1



(a) (b) (c) (d)

Fig. 3. Plot of u (x, t) w.r.t x and t for example 4.3 at (a) α = 0.25 (b) α = 0.5 (c) α = 0.75 (d) α = 1

5. Conclusion:

In this paper, we achieve the exact solution of time fractional nonlinear gas dynamics equations successfully using reduced differential transform through fractional complex transform. This is one of the alternative approach and one can also apply this technique to other nonlinear problems.

References:

[1] Mohammed Tamsir, 'Vineet K. Srivastava, Revisiting the approximate analytical solution of fractional-order gas dynamics equation', *Alexandria Engineering Journal*, Vol.55, pp.867-874, 2016.

[2] S.Das and R.Kumar, 'Approximate analytical solutions of fractional gas dynamics equations', *Applied Mathematics and Computation*, Vol.217, pp.9905-9915, 2011.

[3] Mohamed S. Al-luhaibi, 'New Iterative Method for Fractional Gas Dynamics and Coupled Burger's Equations', *The scientific world journal*, http://dx.doi.org/10.1155/2015/153124.

[4] Hussein Aminikhah and Ali Jamalian, 'Numerical Approximation for Nonlinear Gas Dynamic Equation', *International Journal of Partial Differential Equations*, http://dx.doi.org/10.1155/2013/846749

[5] Ji-Huan He, S.K.Elagan and Z.B.Li, 'Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus,' *Physics Letters A*, Vol. 376, pp.257-259, 2012.

[6] Guy Jumarie, 'Fractional partial differential equations and modified Riemann-Liuville derivatives new methods for solution', *J.Appl.Math.& Comp*, Vol 24, pp.31-48, 2007.

[7] Zeng Biao Li and Ji-Juan He, 'Fractional complex transform for fractional differential equation', *Mathematical and Computational Applications*, Vol.15, pp.970-973, 2010.

[8] S.Saha Ray and S.Sahoo, 'A Novel Analytical Method with fractional complex transform for new exact solutions of time-fractional fifth-order Swada-Kotera Equation,' *Reports on Mathematical Physics*, vol.75, pp. 65-72, 2015.

[9] Keskin Y,Galip Oturanc, ' Reduced differential transform method for partial differential equations,' *Int. Journal of nonlinear sciences and numerical simulation*, Vol.10(6), pp.741-749, 2009.

[10] Keskin Y, Galip Oturanc, 'Reduced differential transform method for generalized KdV equations', *Math.Comput. Appl.* Vol.15 (3), pp.382-393, 2010.

[11] Keskin Y, Ph.D Thesis, Selcuk University, 2010 (in Turkish).

[12] Sunil Kumar, Huseyin Kocak and Ahmet Yıldırım, 'A Fractional Model of Gas Dynamics Equations and its Analytical Approximate Solution Using Laplace Transform', *Verlag der Zeitschrift fur Naturforschung*, pp.389 – 396, 2012.

[13] T.G.Elizarova, 'Quasi-gas dynamics equations', ISBN: 978-3-642-00291-5, Springer Verlag (2009).

[14] S.Weerakoon, 'Application of Sumudu transform to partial differential equations,' *International journal of Mathematical Education in science and technology*, Vol.25 (2), pp.277-283,1994.

[15] H.Jafari, C.Chun, S.Seifi and M.Saeidy, 'Analytical solution for nonlinear gas dynamics equation by homotopy analysis method,*'Appl.Appl.Math*. Vol.4(1), pp.149-154, 2009.

[16] Mohammad Mehdi Rashidi, Sunil kumar, Deepak Kumar and Navid Freidoonimehr, 'New Analytical method for gas dynamics equation arising in shock fronts,' *Computer Physics Communications*, http://dx.doi.org/10.1016/j.cpc.2014.03.025.

[17] Jagadev Singh, Devendra Kumar and Kilicman. A, 'Homotopy Perturbation Method for Fractional Gas Dynamics Equation Using Sumudu Transform,' *Abstract and Applied Analysis*, http://dx.doi.org/10.1155/2013/934060.

[18] Sunil kumar, Cocak. H, Ahmet Yildirim, 'A fractional model for Gas Dynamics Equations and its analytical Approximate Solution Using Laplace Transform,' *Z.Naturforschung*, 67a, pp.389-396, 2012.

[19] D.JEvans, and H.Bulut, A new approach to the gas dynamics equation: an application of the decomposition method, *International Journal of Computer Mathematics*, Vol.79 (7), pp.817-822, 2002.

[20] A.D.Polyanin, Zaitsev V.F, 'Handbook of nonlinear partial differential equations, ISBN-971584883555, *CRC Press, Taylor and Francis Group* (2004).

[21] T.E. Bavisha, 2Ms.M.MadlinAsha, "A Keyword Based User Privacy-Preservation And Copy-Deterrence Scheme For Image Retrieval In Cloud", International Journal of Innovations in Scientific and Engineering Research (IJISER), ISSN: 2347-971X (online), ISSN: 2347-9728(print), Vol.4 (1), pp.30-35, 2017, http://www.ijiser.com/.