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On the Gracefulness of Cycle related graphs

S.Venkatesh[1] and S. Sivagurunathan[2]
[1]Department of Maths, SASTRA University,
Srinivasa Ramanujan Centre, Kumbakonam-612001, India.
mailvenkat1973@gmail.com
[2]Department of Computer Science, SASTRA University,
Srinivasa Ramanujan Centre, Kumbakonam-612001, India.
sivagurunathan@src.sastra.edu

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Abstract

Let $C_n : v_1v_2v_3 \ldots v_nv_1$ be a cycle of length n. A cycle with a C_k- chord, $C_{n,k}$ is the graph obtained from C_n by adding a cycle C_k of length k between the non adjacent vertices v_2 and v_n. A cycle with parallel C_k- chord, $C^+_{n,k}$, is the graph obtained from a cycle C_n by adding a cycle C_k of length k between every pair of non-adjacent vertices $(v_2, v_n), (v_3, v_{n-1}), \ldots, (v_a, v_b)$ where $a = \lfloor \frac{n}{2} \rfloor$, $b = \lfloor \frac{n}{2} \rfloor + 2$, if n is even and $a = \lfloor \frac{n}{2} \rfloor$, $b = \lfloor \frac{n}{2} \rfloor + 3$, if n is odd. In this paper we prove that $C_{n,4}$ and $C^+_{n,4}$ is graceful for $n \equiv 0$ (mod 4) and $C^+_{n,6}$ is graceful for all odd values of $n \geq 5$.

AMS Subject Classification: 05C78

Key Words: Cycle, Cycle with a C_k chord, Cycle with parallel C_k chord, Graph labeling, Graceful Labeling.

1 Introduction

Much interest towards the concept of graph labeling originates from the paper by Rosa in 1967. Rosa [6] introduced graceful labeling as a tool to decompose the complete graph K_{2m+1} into copies of a given tree on m edges. A function f is called a graceful labeling of a graph $G(V,E)$ with m edges, if f is an injection from $V(G)$ to the set $\{0,1,2,\ldots,m\}$ such that when each edge uv is assigned the label $|f(u)-f(v)|$ the resulting edge labels are distinct.

Rosa [6] proved that the cycle C_n is graceful if and only if $n \equiv 0,3$ $(mod\ 4)$. Gracefulness of cycle related graphs are in focus for many years. In 1977, Bodendiek, Schumacher, and H. Wegner [1] conjectured that the cycle with a chord is graceful and verified some special cases. Delorme [2] has proved this result completely. In 1985, Koh and Yap [4] introduced the concept of cycle with a P_k chord and verified the result for $k=3$ and conjectured the general case. Later it is proved by Punnim and Pabhapote [5] in 1987 for all $k \geq 4$. In 2005, Sethuraman and Elumalai [7] defined a cycle with parallel P_k chords as a graph obtained from a cycle C_n for $n \geq 6$ by adding disjoint paths P_k for $k \geq 3$, between each pair of nonadjacent vertices and verified the case for $k = 3,4,6,8$ and 10. For exhaustive results refer the survey by Gallian [3].

Definition. 1.1. A chord of a cycle is an edge joining two non-adjacent vertices of the given cycle.

Definition. 1.2. Let $C_n : v_1v_2v_3\ldots v_nv_1$ be a cycle of length n. A cycle with a C_k- chord, $C_{n,k}$ is the graph is obtained from C_n by adding a cycle C_k of length k between the non adjacent vertices v_2 and v_n. Refer figure.1(a).

Definition. 1.3. A cycle with parallel C_k- chord, $C^+_{n,k}$, is the graph obtained from a cycle C_n by adding a cycle C_k of length k between every pair of non-adjacent vertices $(v_2,v_n),(v_3,v_{n-1}),\ldots,(v_a,v_b)$ where $a = \lfloor\frac{n}{2}\rfloor$, $b = \lfloor\frac{n}{2}\rfloor + 2$, if n is even and $a = \lfloor\frac{n}{2}\rfloor$, $b = \lfloor\frac{n}{2}\rfloor + 3$, if n is odd. Refer figure.1(b).

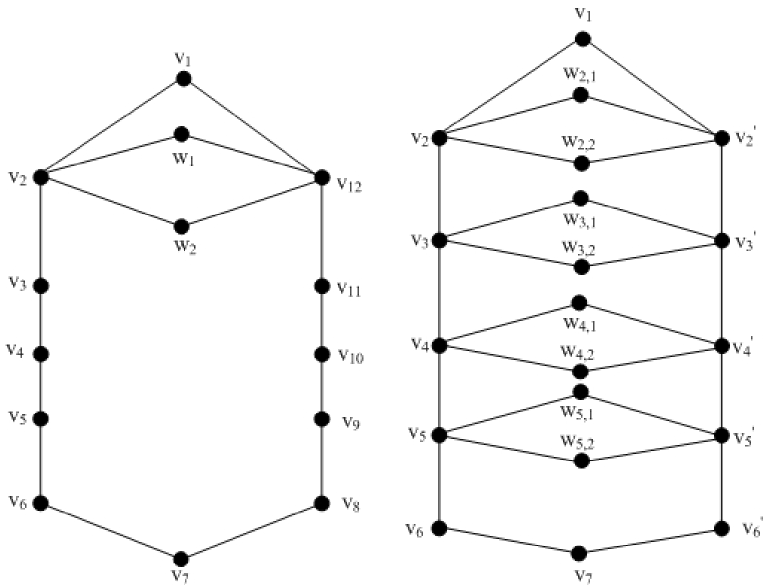

Figure 1: (a) The graph $C_{12,4}$ (b) The graph $C^{+}_{14,4}$

In this paper, we prove that $C_{n,4}$ and $C^{+}_{n,4}$ is graceful for $n \equiv 0$ (mod 4) and $C^{+}_{n,6}$ is graceful for all odd values of $n \geq 5$.

2 Gracefulness of a Cycle with a C_4- chord

Consider a cycle $C_m : v_1v_2v_3 \ldots v_mv_1$ of length m. The graph cycle with a C_4 chord is obtained by adding a cycle $C_4 : v_2w_1v_nw_2v_2$ between the vertices v_2 and v_n. Denote the resulting graph as $C_{m,4}$.

In the following theorem we prove that $C_{m,4}$ is graceful for $m \equiv 0$ (mod 4).

Theorem 1. *Cycle with a C_4- chord, $C_{m,4}$ is graceful, for $m \equiv 0$ (mod 4).*

Proof. Consider a cycle $C_m : v_1v_2v_3 \ldots v_mv_1$ of length m. The graph cycle with a C_4 chord is obtained by adding a cycle C_4 :

$v_2w_1v_nw_2v_2$ between the vertices v_2 and v_n respectively. Let G represents the graph $C_{m,4}$ with $m = 4k$, for $k \geq 1$. We observe that G has $p = m + 2$ vertices and $q = m + 4$ edges.

We label the vertices of the given graph G as follows,

Case 1. When $m = 4$, then G is $C_{4,4}$ and its graceful labeling is illustrated in Figure.2.

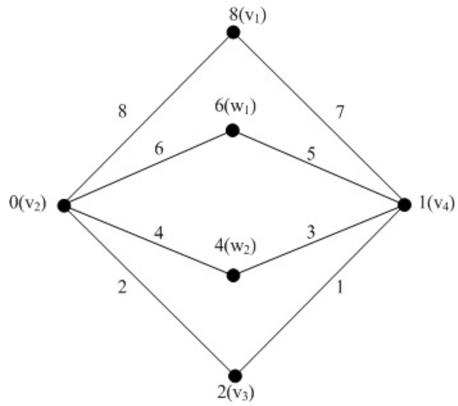

Figure 2: Graceful labeling of the graph $C_{4,4}$

Case 2. When $m = 4k$, for $k \geq 2$.

Define $f(v_1) = q$ and $f(v_m) = 0$.
$f(w_i) = q - 2i$, for $1 \leq i \leq 2$

$$f(v_i) = \begin{cases} m - (\frac{i-1}{2}), & \text{if } 1 \leq i \leq \frac{m}{2} - 1, \ i\text{-- odd} \\ (m-1) - (\frac{i-1}{2}), & \text{if } \frac{m}{2} + 1 \leq i \leq m - 1, \ i\text{-- odd} \\ (\frac{i}{2}), & 2 \leq i \leq m - 2, \ i\text{-- even} \end{cases}$$

From the above vertex labeling we observe that f is an injection from the vertex set of G to the set $\{0, 1, 2, \ldots, q\}$ and the resulting edge labels are distinct from 1 to q. Hence G is graceful. Refer Figure.3. $\square$

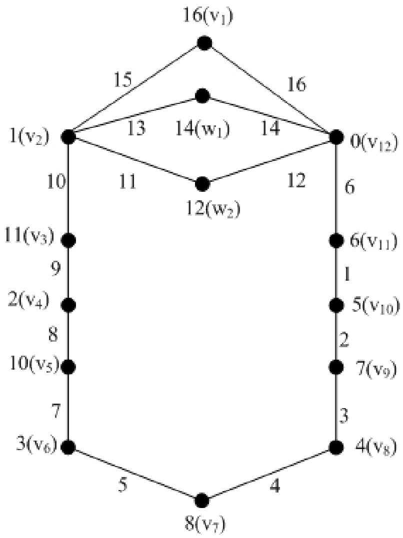

Figure 3: Graceful labeling of the $C_{12,4}$

3 Gracefulness of a cycle with parallel C_k- chords

Consider a cycle $C_m : v_1v_2v_3 \ldots v_{\frac{m}{2}}\ v_{\frac{m}{2}+1}\ v'_{\frac{m}{2}}\ v'_{\frac{m}{2}-1}\ v'_{\frac{m}{2}-2} \ldots v'_3\ v'_2$ v_1 of length m with $m \equiv 0 \pmod 4$. The graph $C^+_{m,4}$ is obtained by adding a cycle $C_4 : v_iw_{i,1}v'_iw_{i,2}v_i$ for $2 \leq i \leq \frac{m}{2}$. The resultant graph is denoted as $C^+_{m,4}$.

Now in the following theorem we prove that $C^+_{m,4}$ is graceful for $m \equiv 0 \pmod 4$.

Theorem 2. *Cycle C_m with a parallel C_4- chord, $C^+_{m,4}$ is graceful for $m \equiv 0$ (mod 4).*

Proof. Consider the graph $G = C^+_{m,4}$ with $m = 4k$, for $k \geq 1$, which is obtained by adding a cycle $C_4 : v_iw_{i,1}v'_iw_{i,2}v_i$ for $2 \leq i \leq \frac{m}{2}$, for $2 \leq i \leq \frac{m}{2}$ with the cycle $C_m : v_1v_2v_3 \ldots v_{\frac{m}{2}}\ v_{\frac{m}{2}+1}\ v'_{\frac{m}{2}}\ v'_{\frac{m}{2}-1}\ v'_{\frac{m}{2}-2}$ $\ldots v'_3\ v'_2\ v_1$. Then G has $p = 2m - 2$ vertices and $q = 3m - 4$ edges and we label the vertices of the given graph G as follows,

Let $f(v_1) = q$.

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For $2 \leq i \leq \frac{m}{2}$ and $i-$ even, define,
$f(v_i) = 3(i-2)$,
$f(v_i^{'}) = 3(i-2)+1$,
$f(w_{i,1}) = 3(m-i)$,
$f(w_{i,2}) = 3(m-i)-2$,

For $3 \leq i \leq \frac{m}{2}-1$ and $i-$ odd, define,
$f(v_i) = q-(3i-2)$,
$f(v_i^{'}) = q-(3i-4)$
$f(w_{i,1}) = 3(i-2)$,
$f(w_{i,2}) = 3(i-2)+1$
Finally, $f(v_{\frac{m}{2}+1}) = (\frac{q}{2})-2$

From the above vertex labeling we observe that f is an injection from the vertex set of G to the set $\{0,1,2,\ldots,q\}$ and the resulting edge labels are distinct from 1 to q. Hence G is graceful. An illustration is given in Figure.4. □

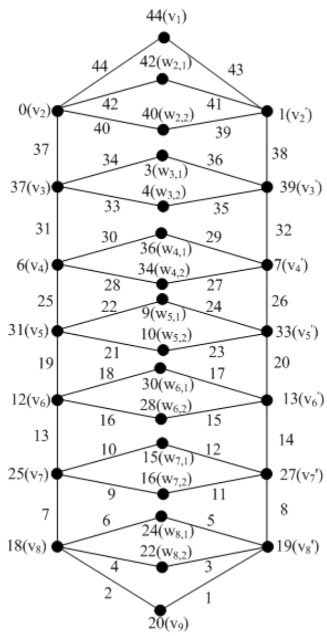

Figure 4: Graceful labeling of the graph $C^{+}_{16,4}$

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For $2 \leq i \leq \frac{m-1}{2}$, the graph $C^+_{m,6}$ is obtained by adding a cycle $C_6 : v_i w_{i,1} v_i^{'} w_{i,2} w_{i,3} w_{i,4} v_i$ with the cycle $C_m : v_1 v_2 v_3 \ldots v_{\frac{m+1}{2}}\ v_{\frac{m+3}{2}}\ v^{'}_{\frac{m-1}{2}}\ v^{'}_{\frac{m-3}{2}} \ldots v_3^{'}\ v_2^{'}\ v_1$ with $m = 2k + 1$ with $k \geq 2$.

Theorem 3. *Cycle C_m with a parallel C_6- chord, $C^+_{m,6}$ is graceful for all odd values of $m \geq 5$.*

Proof. Consider the graph $G = C^+_{m,6}$, where $m = 2k + 1$ with $k \geq 2$. Then for $2 \leq i \leq \frac{m-1}{2}$, G is obtained by adding the cycle $C_6 : v_i w_{i,1} v_i^{'} w_{i,2} w_{i,3} w_{i,4} v_i$ with $C_m : v_1 v_2 v_3 \ldots v_{\frac{m+1}{2}}\ v_{\frac{m+3}{2}}\ v^{'}_{\frac{m-1}{2}}\ v^{'}_{\frac{m-3}{2}} \ldots v_3^{'}\ v_2^{'}\ v_1$.

We observe that G has $p = 3m - 6$ vertices and $q = 4m - 9$ edges and we label the vertices of the given graph G as follows,

Case 1. When $m = 5$, then G is $C^+_{5,6}$ and its gracefulness is illustrated in Figure.5.

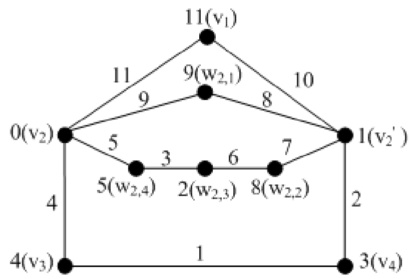

Figure 5: Graceful labeling of the graph $C^+_{5,6}$

Case 2. When $m = 2k + 1$, for $k \geq 3$.
Let $f(v_1) = q$.

For $2 \leq i \leq \lfloor \frac{m}{2} \rfloor$ and $i-$ even, define,
$f(v_i) = 4(i - 2)$, $f(v_i^{'}) = 4(i - 2) + 1$,
$f(w_{i,1}) = q - 4i + 6$, $f(w_{i,2}) = q - 4i + 5$,
$f(w_{i,3}) = 4i - 6$, $f(w_{i,4}) = q - 4i + 2$,

For $3 \leq i \leq \lfloor \frac{m}{2} \rfloor$ and $i-$odd, define,
$f(v_i) = q - 4i + 5$, $f(v_i^{'}) = q - 4i + 4$,
$f(w_{i,1}) = 4i - 9$, $f(w_{i,2}) = 4i - 8$,
$f(w_{i,3}) = q - 4i + 3$, $f(w_{i,4}) = 4i - 5$.

Let $r = \frac{n}{2}$, then define,

$$f(v_{\lceil r \rceil}) = \begin{cases} f(w_{\lfloor r \rfloor,4}) + 1, & \text{if } m = 4k+3,\ k \geq 1 \\ f(w_{\lfloor r \rfloor,4}) - 1, & \text{if } m = 4k+1,\ k \geq 1 \end{cases}$$

$$f(v_{\lceil r \rceil+1}) = \begin{cases} f(w_{\lfloor r \rfloor,4}) + 2, & \text{if } m = 4k+3,\ k \geq 1 \\ f(w_{\lfloor r \rfloor,4}) - 2, & \text{if } m = 4k+1,\ k \geq 1 \end{cases}$$

From the above vertex labeling we observe that f is an injection from the vertex set of G to the set $\{0, 1, 2, ..., q\}$ and the resulting edge labels are distinct from 1 to q. Hence G is graceful. An illustration is given in Figure.6. □

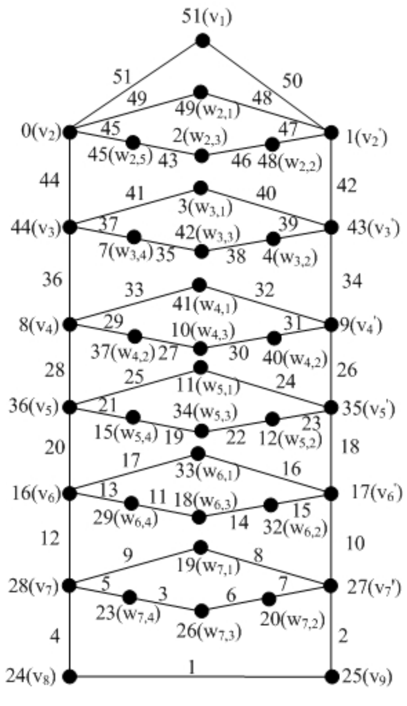

Figure 6: Graceful labeling of the graph $C^{+}_{15,6}$

4 Discussion

It is proved that the graphs $C_{n,4}$, $C^+_{n,4}$ (for $n = 4k$, $k \geq 1$) and $C^+_{n,6}$ ($n-$odd) admits graceful labeling. However I strongly feel that $C_{n,m}$ admits graceful labeling for all values of $n = 2k$ and $m \equiv 0(\text{mod } 4)$. Further is it true that cycle with parallel chords $C^+_{n,k}$ is graceful for all values of n with $k = 2m$, for $m \geq 2$.

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