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Signed Edge Domination on Rooted Product Graph

[1]C. Shobha Rani, [2]S. Jeelani Begum and [3]G.S.S. Raju

[1]Department of Mathematics,

Madanapalle Institute of Technology & Science,

Madanapalle, India.

charapallishobha@gmail.com

[2]Department of Mathematics,

Madanapalle Institute of Technology & Science,

Madanapalle, India.

sjbmaths@gmail.com

[3]Department of Mathematics,

JNTUA College of Engineering,

Pulivendula, India

rajugss@yahoo.com

Abstract

Let G be a rooted product graph of path with a cycle graph with the vertex set V and the edge set E. Here P_n be a Path graph with n vertices and $C_m (m \geq 3)$ be a cycle with a sequence of n rooted graphs $C_{m1}, C_{m2}, C_{m3}, ---, C_{mn}$. Then by $P_n(C_m)$ we denote the graph obtained by identifying the root of C_{mi} with the ith vertex of P_n. We call $P_n(C_m)$ the rooted product of P_n by C_m and it is denoted by $P_n \circ C_m$. Every ith vertex of P_n is merging with any one vertex in every ith copy of C_m. So in $G = P_n \circ C_m$, P_n contains n vertices and C_m contains (m-1) vertices in each copy of C_m. In this paper we discuss some results on rooted product graph of path with a cycle graph.

Key Words: Rooted product graph, signed dominating functions, signed domination number.

1. Introduction

Graph theory is an important subject in mathematics. Applications in many fields like coding theory, Logical Algebra, Engineering communications and Computer networking. The rooted product graphs are used in internet systems for connecting internet to one system to other systems. Mostly Product of graphs used in discrete mathematics. In 1978, Godsil and McKay [3] introduced a new product on two graphs G_1 and G_2, called rooted product denoted by $G_1 \circ G_2$. In 1977, Mitchell and Hedetniemi [7] have studied about "Edge domination in trees". In 2001, Xu [2] have studied about "On signed edge domination numbers of graphs". Further we studied about signed edge domination in [1, 4, 5, 6]. Here we can find out signed edge domination related parameters on rooted product graph.

2. Results on Signed edge domination

Theorem 2.1: If m is divisible by 3 then the signed edge domination number of $\mathrm{G} = P_{\mathrm{n}} \circ C_m$ is $\gamma_s(G) = n\left[m - \frac{2m}{3} + 1\right] - 1$.

Proof: Let $\mathrm{G} = P_{\mathrm{n}} \circ C_m$ be a rooted product graph and m=3k. Where k is a natural number set.

We define a signed edge dominating function $f : E \to [0,1]$ as follows:

$$f(e) = \begin{cases} -1, \text{for } \frac{m}{3} \text{ edges in each copy of } \mathrm{C}_m \text{ in G,} \\ +1, \text{otherwise.} \end{cases}$$

Then by the definition of the function.

$$f(e_1) = f(e_2) = --- = f(e_{n-1}) = 1,$$
$$f(h_{ij}) = -1, \text{ if } \mathrm{j} \equiv 1 (\text{mod } 3) \text{ in each copy } C_m \text{ of } G,$$
$$f(h_{ij}) = 1, \text{ otherwise.}$$

By the function definition, the values -1 is assigned to $\frac{m}{3}$ edges in each copy of $\mathrm{C_m}$ and +1 is assigned to remaining vertices in G.

Case 1: If $e_i \in P_{\mathrm{n}}$, where $i = 1, 2, ---, (n-1)$.

If $adj(e_i) = 5$ then $\sum_{e \in N[e_i]} f(e) = 1+1+1+1+(-1)+(-1) = 2$.

If $adj(e_i) = 6$ then $\sum_{e \in N[e_i]} f(e) = 1+1+1+1+(-1)+(-1)+1 = 3$.

Case 2: If $h_{ij} \in C_m; i = 1, 2, ---, n; j = 1, 2, 3, ---, m$.

Subcase 1: Suppose $adj(h_{ij})=2$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are no edges of P_n and two edges of C_m and there are two edges which are drawn from the vertices u_{ij} and $u_{i(j+1)}$ of C_m.

Therefore $\sum_{e\in N[h_{ij}]} f(e)=1+(-1)+1=1$.

Subcase 2: Suppose $adj(h_{ij})=3$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are two edges of C_m, one edge of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i=1 or\, n$.

Therefore $\sum_{e\in N[h_{ij}]} f(e)=(-1)+1+1+1=2$.

Subcase 3: Suppose $adj(h_{ij})=4$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are two edges of C_m, two edges of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i=1 or\, n$.

Therefore $\sum_{e\in N[h_{ij}]} f(e)=\left[1+(-1)+1\right]+1+1=3$.

From the above possible cases, we get $\sum_{e\in E(G)} f(e)\geq 1$.

This implies f is a signed edge dominating function.

Now the minimality check for of f. Define another function $g:E\to\{-1,1\}$ by

$$g(e)=\begin{cases}-1, \text{for } \frac{m}{3} \text{ edges in each copy of } C_m \text{ in G,}\\ -1, \text{if } e=e_k\in P_n \text{ for some k,}\\ +1, \text{otherwise.}\end{cases}$$

Since strict equality not holds at an edge $e_i\in P_n$, it follows that g<f.

Case 1: If $e_i\in P_n$, where $i=1,2,---,(n-1)$.

Sub case 1: Let $e_k\in N[e_i]$.

If $adj(e_i)=5$ then $\sum_{e\in N[e_i]} g(e)=1+(-1)+\underbrace{1+(-1)}_{2-times}=0$.

If $adj(e_i)=6$ then $\sum_{e\in N[e_i]} g(e)=1+(-1)+1+\underbrace{(-1)+1}_{2-times}=1$.

Sub case 2: Let $e_k\notin N[e_i]$.

If $adj(e_i) = 5$ then $\sum_{e \in N[e_i]} g(e) = 1 + 1 + \underbrace{1 + (-1)}_{2-times} = 2$.

If $adj(e_i) = 6$ then $\sum_{e \in N[e_i]} g(e) = 1 + 1 + 1 + \underbrace{(-1) + 1}_{2-times} = 3$.

Case 2: If $h_{ij} \in C_m; i = 1, 2, ---, n; j = 1, 2, 3, ---, m$.

Subcase 1: Suppose $adj(h_{ij}) = 2$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are no edges of P_n and two edges of C_m and there are two edges which are drawn from the vertices u_{ij} and $u_{i(j+1)}$ of C_m.

Therefore $\sum_{e \in N[h_{ij}]} g(e) = 1 + (-1) + 1 = 1$.

Subcase 2: Suppose $adj(h_{ij}) = 3$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are two edges of C_m, one edge of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i = 1 or\, n$.

Let $e_k \in N[h_{ij}]$ then $\sum_{e \in N[h_{ij}]} g(e) = (-1) + 1 + 1 + (-1) = 0$.

Let $e_k \notin N[h_{ij}]$ then $\sum_{e \in N[h_{ij}]} g(e) = (-1) + 1 + 1 + 1 = 2$.

Subcase 3: Suppose $adj(h_{ij}) = 4$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are two edges of C_m, two edges of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i = 1 or\, n$.

Therefore $\sum_{e \in N[h_{ij}]} g(e) = \begin{cases} 1 + (-1) + 1 + (-1) + 1 = 1, & \text{if } e_k \in N[h_{ij}] \\ 1 + (-1) + 1 + 1 + 1 = 3, & \text{if } e_k \notin N[h_{ij}] \end{cases}$.

From the above possible cases, we get

$\sum_{e \in E(G)} g(e) < 1, \text{ for some } e \in E$.

This implies g is not a signed edge dominating function.

Hence f is a minimal signed edge dominating function.

Now signed edge domination number is

$$\sum_{e \in E(G)} f(e) = \underbrace{\left(\frac{m}{3}\right)(-1) + \left(m - \frac{m}{3}\right)(+1)}_{n-times} + (n-1) = n\left[m - \frac{2m}{3} + 1\right] - 1.$$

Theorem 2.2: If m is not divisible by 3, that is m=3k+1 then the signed edge domination number of $G = P_n \circ C_m$ is

$\gamma_s(G) = n\left[m - 2\left\lfloor \frac{m}{3} \right\rfloor + 1\right] - 1$.

Proof: Let $G = P_n \circ C_m$ be a rooted product graph and m=3k+1. Where k is a natural number set.

We define a signed edge dominating function $f : E \to [0,1]$ as follows:

$$f(e) = \begin{cases} -1, \text{for } \frac{m}{3} \text{ edges in each copy of } C_m \text{ in G}, \\ +1, \text{otherwise.} \end{cases}$$

Then by the definition of the function.

$f(e_1) = f(e_2) = --- = f(e_{n-1}) = 1,$
$f(h_{ij}) = -1, \text{ if } j \equiv 0 (\text{mod } 3) \text{ in each copy } C_m \text{ of } G,$
$f(h_{ij}) = 1, \text{ otherwise.}$

By the function definition, the values -1 is assigned to $\frac{m}{3}$ edges in each copy of C_m and +1 is assigned to remaining vertices in G.

Case 1: If $e_i \in P_n$, where $i = 1, 2, ---, (n-1)$.

If $adj(e_i) = 5$ then $\sum_{e \in N[e_i]} f(e) = 1+1+1+1+1+1 = 6$.

If $adj(e_i) = 6$ then $\sum_{e \in N[e_i]} f(e) = 1+1+1+1+1+1+1 = 7$.

Case 2: If $h_{ij} \in C_m; i = 1, 2, ---, n; j = 1, 2, 3, ---, m$.

Subcase 1: Suppose $adj(h_{ij}) = 2$, $N[h_{ij}]$, j=1,2,3,---,m there are no edges of P_n and two edges of C_m and there are two edges which are drawn from the vertices u_{ij} and $u_{i(j+1)}$ of C_m.

Therefore $\sum_{e \in N[h_{ij}]} f(e) = 1 + (-1) + 1 = 1$.

Subcase 2: Suppose $adj(h_{ij}) = 3$, $N[h_{ij}]$, j=1,2,3,---,m there are two edges of C_m, one edge of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i = 1 \text{ or } n$.

Therefore $\sum_{e \in N[h_{ij}]} f(e) = \begin{cases} (-1)+1+1+1 = 2, \text{if } -1 \in N[h_{ij}] \\ 1+1+1+1 = 4, \text{if } -1 \notin N[h_{ij}] \end{cases}$.

Subcase 3: Suppose $adj(h_{ij}) = 4$, $N[h_{ij}]$, j=1,2,3,---,m there are two edges of C_m, two edges of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i=1 or\, n$.

Therefore $\sum_{e\in N[h_{ij}]} f(e) = \begin{cases}(-1)+1+1+1+1=3, if\ -1\in N[h_{ij}]\\ 1+1+1+1+1=5, if\ -1\notin N[h_{ij}]\end{cases}$.

From the above possible cases, we get $\sum_{e\in E(G)} f(e)\geq 1$.

This implies f is a signed edge dominating function.

Now the minimality check for of f. Define another function $g: E\rightarrow\{-1,1\}$ by

$$g(e)=\begin{cases}-1, \text{for } \frac{m}{3} \text{ edges in each copy of } C_m \text{ in G,}\\ -1, \text{if e=e}_k\in P_n \text{ for some k,}\\ +1, \text{otherwise.}\end{cases}$$

Since strict equality not holds at an edge $e_i\in P_n$, it follows that g<f.

Case 1: If $e_i\in P_n$, where $i=1,2,---,(n-1)$.

Sub case 1: Let $e_k\in N[e_i]$.

If $adj(e_i)=5$ then $\sum_{e\in N[e_i]} g(e) = 1+(-1)+\underbrace{1+1}_{2-times}=4$.

If $adj(e_i)=6$ then $\sum_{e\in N[e_i]} g(e) = 1+(-1)+1+\underbrace{1+1}_{2-times}=5$.

Sub case 2: Let $e_k\notin N[e_i]$.

If $adj(e_i)=5$ then $\sum_{e\in N[e_i]} g(e) = 1+1+\underbrace{1+1}_{2-times}=6$.

If $adj(e_i)=6$ then $\sum_{e\in N[e_i]} g(e) = 1+1+1+\underbrace{1+1}_{2-times}=7$.

Case 2: If $h_{ij}\in C_m; i=1,2,---,n;\ j=1,2,3,---,m$.

Subcase 1: Suppose $adj(h_{ij})=2$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are no edges of P_n and two edges of C_m and there are two edges which are drawn from the vertices u_{ij} and $u_{i(j+1)}$ of C_m.

Therefore $\sum_{e\in N[h_{ij}]} g(e) = 1+(-1)+1=1$.

Subcase 2: Suppose $adj(h_{ij})=3$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are two edges of C_m, one edge of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i=1 or\, n$.

Let $e_k \in N[h_{ij}] \Rightarrow \sum_{e\in N[h_{ij}]} g(e) = \begin{cases} (-1)+1+1+(-1)=0, if\ -1\in N[h_{ij}] \\ 1+1+1+(-1)=2, if\ -1\notin N[h_{ij}] \end{cases}$.

Let $e_k \notin N[h_{ij}] \Rightarrow \sum_{e\in N[h_{ij}]} g(e) = \begin{cases} (-1)+1+1+1=2, if\ -1\in N[h_{ij}] \\ 1+1+1+1=4, if\ -1\notin N[h_{ij}] \end{cases}$.

Subcase 3: Suppose $adj(h_{ij})=4$, N$\left[h_{ij}\right]$, j=1,2,3,---,m there are two edges of C_m, two edges of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i=1 or\ n$.

Let $e_k \in N[h_{ij}] \Rightarrow \sum_{e\in N[h_{ij}]} g(e) = \begin{cases} (-1)+1+1+1+(-1)=1, if\ -1\in N[h_{ij}] \\ 1+1+1+1+(-1)=3, if\ -1\notin N[h_{ij}] \end{cases}$.

Let $e_k \notin N[h_{ij}] \Rightarrow \sum_{e\in N[h_{ij}]} g(e) = \begin{cases} (-1)+1+1+1+1=3, if\ -1\in N[h_{ij}] \\ 1+1+1+1+1=5, if\ -1\notin N[h_{ij}] \end{cases}$.

From the above possible cases, we get

$\sum_{e\in E(G)} g(e) < 1,$ for some $e \in E$.

This implies g is not a signed edge dominating function.

Hence f is a minimal signed edge dominating function, if m=3k+1.

Now signed edge domination number is

$$\sum_{e\in E(G)} f(e) = \underbrace{\left(\left\lfloor \frac{m}{3} \right\rfloor\right)(-1) + \left(m - \left\lfloor \frac{m}{3} \right\rfloor\right)(+1)}_{n-times} + (n-1) = n\left[m - 2\left\lfloor \frac{m}{3} \right\rfloor + 1\right] - 1.$$

Theorem 2.3: If m is not divisible by 3, that is m=3k+2 then the function $f : E \to [0,1]$ is defined by

$$f(e) = \begin{cases} -1, \text{for } \frac{m}{3} \text{ edges in each copy of } C_m \text{ in G}, \\ +1, \text{otherwise.} \end{cases}$$

It becomes not a minimal signed edge dominating function of $G = P_n \circ C_m$.

Proof: Let $G = P_n \circ C_m$ be a rooted product graph and m=3k+2. Where k is a natural number set.

We define a signed edge dominating function as in the hypothesis.

Then by the definition of the function.

$f(e_1) = f(e_2) = --- = f(e_{n-1}) = 1,$
$f(h_{ij}) = -1,$ if $j \equiv 0 (\text{mod } 3)$ in each copy C_m of G,
$f(h_{ij}) = 1,$ otherwise.

By the function definition, the values -1 is assigned to $\frac{m}{3}$ edges in each copy of C_m and +1 is assigned to remaining vertices in G.

Case 1: If $e_i \in P_n$, where $i = 1, 2, ---, (n-1)$.

If $adj(e_i) = 5$ then $\sum_{e \in N[e_i]} f(e) = 1+1+1+1+1+1 = 6$.

If $adj(e_i) = 6$ then $\sum_{e \in N[e_i]} f(e) = 1+1+1+1+1+1+1 = 7$.

Case 2: If $h_{ij} \in C_m; i = 1, 2, ---, n; j = 1, 2, 3, ---, m$.

Subcase 1: Suppose $adj(h_{ij}) = 2$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are no edges of P_n and two edges of C_m and there are two edges which are drawn from the vertices u_{ij} and $u_{i(j+1)}$ of C_m.

Therefore $\sum_{e \in N[h_{ij}]} f(e) = 1 + (-1) + 1 = 1$.

Subcase 2: Suppose $adj(h_{ij}) = 3$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are two edges of C_m, one edge of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i = 1 \, or \, n$.

Therefore $\sum_{e \in N[h_{ij}]} f(e) = 1+1+1+1 = 4$.

Subcase 3: Suppose $adj(h_{ij}) = 4$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are two edges of C_m, two edges of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i = 1 \, or \, n$.

Therefore $\sum_{e \in N[h_{ij}]} f(e) = 1+1+1+1+1 = 5$.

From the above possible cases, we get $\sum_{e \in E(G)} f(e) \geq 1$.

This implies f is a signed edge dominating function.

Now minimality check for of f. Define another function $g : E \to \{-1, 1\}$ by

$$g(e) = \begin{cases} -1, \text{for } \frac{m}{3} \text{ edges in each copy of } C_m \text{ in G,} \\ -1, \text{if } e = e_k \in P_n \text{ for some k,} \\ +1, \text{otherwise.} \end{cases}$$

Since strict equality not holds at an edge $e_i \in P_n$, it follows that g<f.

Case 1: If $e_i \in P_n$, where $i = 1, 2, ---, (n-1)$.

Sub case 1: Let $e_k \in N[e_i]$.

If $adj(e_i) = 5$ then $\sum_{e \in N[e_i]} g(e) = 1 + (-1) + \underbrace{1+1}_{2-times} = 4$.

If $adj(e_i) = 6$ then $\sum_{e \in N[e_i]} g(e) = 1 + (-1) + 1 + \underbrace{1+1}_{2-times} = 5$.

Sub case 2: Let $e_k \notin N[e_i]$.

If $adj(e_i) = 5$ then $\sum_{e \in N[e_i]} g(e) = 1 + 1 + \underbrace{1+1}_{2-times} = 6$.

If $adj(e_i) = 6$ then $\sum_{e \in N[e_i]} g(e) = 1 + 1 + 1 + \underbrace{1+1}_{2-times} = 7$.

Case 2: If $h_{ij} \in C_m; i = 1, 2, ---, n; j = 1, 2, 3, ---, m$.

Subcase 1: Suppose $adj(h_{ij}) = 2$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are no edges of P_n and two edges of C_m and there are two edges which are drawn from the vertices u_{ij} and $u_{i(j+1)}$ of C_m.

Therefore $\sum_{e \in N[h_{ij}]} g(e) = 1 + (-1) + 1 = 1$.

Subcase 2: Suppose $adj(h_{ij}) = 3$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are two edges of C_m, one edge of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i = 1 \, or \, n$.

Let $e_k \in N[h_{ij}]$ then $\sum_{e \in N[h_{ij}]} g(e) = 1 + 1 + 1 + (-1) = 2$.

Let $e_k \notin N[h_{ij}]$ then $\sum_{e \in N[h_{ij}]} g(e) = 1 + 1 + 1 + 1 = 4$.

Subcase 3: Suppose $adj(h_{ij}) = 4$, $N\left[h_{ij}\right]$, j=1,2,3,---,m there are two edges of C_m, two edges of P_n and there is an edge which are drawn from the vertices

u_{ij},i=1,2,---,n;j=1 or (m-1) and $v_i, i = 1 \, or \, n$.

Let $e_k \in N[h_{ij}]$ then $\sum_{e \in N[h_{ij}]} g(e) = 1 + 1 + 1 + 1 + (-1) = 3$.

Let $e_k \notin N[h_{ij}]$ then $\sum_{e \in N[h_{ij}]} g(e) = 1 + 1 + 1 + 1 + 1 = 5$.

From the above possible cases, we get $\sum_{e \in E(G)} g(e) \geq 1.$.

This implies g is also a signed edge dominating function.

Hence f is not a minimal signed edge dominating function, if m=3k+2.

References

[1] Zelinka B., On signed edge domination numbers of trees, Mathematica Bohemica 127(1) (2002), 49-55.

[2] Xu B., On signed edge domination numbers of graphs, Discrete Mathematics 239(1-3) (2001), 179-189.

[3] Godsil C.D., McKay B.D., A new graph product and its spectrum, Bulletin of the Australian Mathematical Society 18(1) (1978), 21-28.

[4] Karami H., Khodkar A., Sheikholeslami S.M., An improved upper bound for signed edge domination numbers in graphs, Utilitas Math 78 (2009), 121–128.

[5] Karami H., Khodkar A., Sheikholeslami S.M., Signed edge domination numbers in trees, Ars Combinatoria 93 (2009), 451-457.

[6] Xia H., Wei F., Xu Chunlei J., Signed edge total domination numbers of two classes of graphs, International Journal of Pure and Applied Mathematics 81(4) (2012), 581-590.

[7] Mitchell S., Hedetniemi S.T., Edge domination in trees, Congr. Numer 19 (1977), 489–509.