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The $2t-$Pebbling Property on the Zig-Zag Chain Graph of n Odd Cycles

A. Lourdusamy[1] and J. Jenifer Steffi[2]

[1,2]*St. Xavier's College (Autonomous), Palayamkottai- 627002, Tamilnadu, India.*
[1]*lourdusamy15@gmail.com*
[2]*jenifersteffi.j@gmail.com*

Abstract

A graph G has the $2t-$pebbling property if for any distribution with more than $2f_t(G) - q$ pebbles, it is possible, using the sequence of pebbling moves, to put $2t$ pebbles on any vertex. In this paper, we show that the zig-zag chain graph of n copies of odd cycles satisfies the two-pebbling property and the $2t-$pebbling property.

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1 Introduction

Throughout this paper, when we speak of a graph G, we consider only simple and connected graphs. For the definition of other graph theoritical terms, we refer to [1]. Frequently, the letter v will be used to denote the specified vertex of the graph under consideration. Let D be a distribution of pebbles on the vertices of G. Let $p(v)$ denote the number of pebbles distributed on the vertex v and let $p(G) = \sum_{v\in V(G)} p(v)$. The operation of pebbling movement is called a *pebbling step*, defined as removing two pebbles from a vertex and adding one on an adjacent vertex. To understand the pebbling concepts, we need the following definitions.

Definition 1. [2] [3] The *pebbling number* of G, $f(G)$, and the *t-pebbling number* of G, $f_t(G)$, are the smallest numbers, such that from any placement of $f(G)$ pebbles or $f_t(G)$ pebbles, it is possible to move one or t pebbles, respectively, to any specified vertex by a sequence of pebbling moves. Thus, $f(G)$ and $f_t(G)$ are the maximum values of $f(G, v)$ and $f_t(G, v)$ over all vertices v.

Definition 2. [2] [4] Let D be a distribution of pebbles on G, let q be the number of vertices with at least one pebble. We say that a graph G has the $2t$-*pebbling property*, if for any distribution with more than $2f_t(G) - q$ pebbles, it is possible to move $2t$ pebbles to any specified vertex. Put $t = 1$, we can get the $2-$pebbling property.

This paper is organized as follows. In Section 2, we give some preliminary results which are used in our main results. In Section 3, we show that the zig-zag chain graph, ZZ_n, of n copies of odd cycles satisfies the two-pebbling property. In Section 4, we study the $2t-$pebbling property of the graph ZZ_n.

2 Preliminaries

In this section, we give the definition of the zig-zag chain graph of n copies of odd cycles and provide some important results which are used in the subsequent sections.

Definition 3. [7] The *zig-zag chain graph of n copies of odd cycles* denoted by ZZ_n, is a graph which consists of zig-zag sequence of n copies odd cycles, C_{2k+1} with $k \geq 2$. We define ZZ_n as follows.
Case (i) When n is even.
The vertex set of ZZ_n is $V(ZZ_n) = \{a_i, b_i : 1 \leq i \leq \frac{n}{2}(2k-1)\} \cup \{x, y\}$
The edge set of ZZ_n is
$$E(ZZ_n) = \{a_ia_{i+1}, b_ib_{i+1} : 1 \leq i \leq \tfrac{n}{2}(2k-1) - 1\} \cup \{xa_1, xb_1, ya_{\frac{n}{2}(2k-1)}, yb_{\frac{n}{2}(2k-1)}\} \cup \{a_{i(2k-1)-(k-1)}b_{i(2k-1)-(k-1)}, a_{j(2k-1)}b_{j(2k-1)+1} : 1 \leq i \leq \tfrac{n}{2}, 1 \leq j \leq \tfrac{n-2}{2}\}$$
Case (ii) When n is odd.
The vertex set of ZZ_n is $V(ZZ_n) = \{a_i, b_i : 1 \leq i \leq \left(nk - \frac{n+1}{2}\right)\} \cup \{x, y, z\}$
The edge set of ZZ_n is
$$E(ZZ_n) = \{a_ia_{i+1}, b_ib_{i+1} : 1 \leq i \leq \left(nk - \tfrac{n+1}{2}\right) - 1\} \cup \{xa_1, xb_1, ya_{\left(nk-\frac{n+1}{2}\right)}, yz, zb_{\left(nk-\frac{n+1}{2}\right)}\} \cup \{a_{i(2k-1)-(k-1)}b_{i(2k-1)-(k-1)}, a_{j(2k-1)}b_{j(2k-1)+1} : 1 \leq i, j \leq \tfrac{n-1}{2}\}.$$

The graph ZZ_n has n copies of C_{2k+1}, and we label each cycle as $A_1, A_2, ...,$ and A_n in order. We state below some results on graph pebbling which will be used in our discussion.

Theorem 4. *[4] Let P_n be the path with n vertices. Then $f_t(P_n) = t2^{n-1}$ and P_n satisfies the $2t-$pebbling property.*

Theorem 5. *[5] [6] Let C_n denote a simple cycle with n vertices, where $n \geq 3$. Then*

1. $$f_t(C_n) = \begin{cases} t2^k, & n=2k \\ \frac{2^{k+2}-(-1)^{k+2}}{3} + (t-1)2^k, & n=2k+1. \end{cases}$$

2. *The graph C_n satisfies the $2t-$pebbling property.*

Theorem 6. *[7] Let ZZ_n be the zig-zag chain graph of n copies of odd cycles. Then the following are true.*
(1)*For $n = 2m$, we have, $f_t(ZZ_{2m}) = t2^{m(2k-1)+1}$.*
(2)*For $n = 2m+1$, we have, $f_t(ZZ_{2m+1}) = t2^{(2m+1)k-m} + (2m+1)k - (m+1)$.*

Theorem 7. *[7] Let ZZ_n be the zig-zag chain graph of n copies of odd cycles. Then the following are true.*
(1)*For $n = 2m$, we can move at least 2^{k-1} pebbles to the vertex $a_{(m-1)(k-1)+mk}$, when $p(\langle ZZ_{2m} - A_{2m}\rangle) \geq 2^{m(2k-1)+1} - (\frac{2^{k+2}-(-1)^{k+2}}{3}) + 1$.*
(2)*For $n = 2m+1$, we can move at least 2^{k-1} pebbles to the vertex $a_{m(2k-1)}$, when $p(\langle ZZ_{2m+1} - A_{2m+1}\rangle) \geq 2^{(2m+1)k-m} + k(2m+1) - (m+1) - (\frac{2^{k+2}-(-1)^{k+2}}{3}) + 1$.*

3 The two-pebbling property

In [7], the authors have computed the pebbling number and the $t-$pebbling number of the zig-zag chain graph of n copies of odd cycles. In this section, we show that the graph ZZ_n satisfies the two-pebbling property.

Theorem 8. *Let $G \cong ZZ_2$ be the zig-zag chain graph of two copies of odd cycles. If $p(G) \geq 2f(G) - q + 1$, then we can move two pebbles to the vertex v.*

Proof. Without loss of generality, assume that $v \in A_2$. Suppose that $p(A_2) \geq (\frac{2^{k+2}-(-1)^{k+2}}{3})$. Then, we can move one pebble to the vertex v. This leaves at least $p(G) - (\frac{2^{k+2}-(-1)^{k+2}}{3})$ pebbles in G. We have to move an additional pebble to the vertex v. Thus, we claim the following:
*Claim*1 : $p(G) - (\frac{2^{k+2}-(-1)^{k+2}}{3}) \geq f(G)$
We have, $p(G) - (\frac{2^{k+2}-(-1)^{k+2}}{3}) \geq 2(2^{2k}) - q + 1 - 2^{k+1}$
$\geq 2^{2k} + 2^{k+1}(2^{k-1} - 1) - 4k + 1$
$> 2^{2k}$. Hence at least 2^{2k} pebbles are retained in G. By Theorem 6, we can move an additional pebble to the vertex v. Now, assume that $p(A_2) < (\frac{2^{k+2}-(-1)^{k+2}}{3})$. Suppose $v \in V(A_2) - \{a_{2k-1}, b_{2k-1}, y\}$. By Claim 1, A_1 has at least 2^{2k} pebbles. So that by Theorem 7, we can move at least 2^{k-1} pebbles to the vertex a_k or to the vertex b_k, by using more than $2^{2k} - (\frac{2^{k+2}-(-1)^{k+2}}{3})$ pebbles. Therefore according to the choice of v, we can transfer two pebbles to the vertex v, either from the vertex a_k or from the vertex b_k. For considering the exceptional cases, we may assume that $v = y$. Let $A : a_1, a_2, ..., y$ and $B : b_1, b_2, ..., y$ be two paths and note that $f(A) = f(B) = 2^{2k-1}$. Let q_A and q_B be the number of occupied vertices in A and B respectively. Suppose any one of these paths satisfies the two-pebbling property. Then we are done. Suppose not, we must have the following inequalities: $\frac{p(x)}{2} + p(A) \geq 2f(A) - q_A + 1$ or $\frac{p(x)}{2} + p(B) \geq 2f(B) - q_B + 1$. Without loss of generality, assume that $\frac{p(x)}{2} + p(A) \geq 2f(A) - q_A + 1$. Then we can move two pebbles to the vertex v. Otherwise, $p(x) + p(A) + p(B) < 2f(G) - q + 1$, which contradicts the total number of pebbles distributed in G. □

Theorem 9. *Let $G \cong ZZ_3$ be the zig-zag chain graph of three copies of odd cycles. Suppose that $p(G) \geq 2f(G) - q + 1$. Then we can move two pebbles to the* *vertex* v.

Proof. Let q_i be the number of occupied vertices in A_i, for $i = 1, 2, 3$. The graph G can be partitioned into two subgraphs say $S_1 \equiv A_1 \cup A_2$ and $S_2 \equiv A_2 \cup A_3$. For convenience, we may take, q_{S_i} is the number of occupied vertices in S_i, $i = 1, 2$. Thus we can write, $q = q_{S_1} + q_{S_2} - q_2$. Clearly, the vertex v belongs to A_i, for some i. We consider the following possibilities:

Case (1) $v \in A_2$. Suppose $p(S_1) \geq 2(2^{2k}) - q_{S_1} + 1$. Then we are done. Otherwise, we have to move two pebbles to the vertex v by using the unused pebbles in S_2. Thus we claim the following:
$Claim2 : p(G) - p(S_1) \geq 2(2^{2k}) - q_{S_2} + 1$
We have, $p(G) - p(S_1) = 2^{3k} + 6k - 2^{2k+1} - 3 - q_{S_2} + q_2$
$\geq 4(2^{2k}) - 2(2^{2k}) - q_{S_2} + 1 + 6k + q_2 - 4$
$> 2(2^{2k}) - q_{S_2} + 1$. Hence by claim, we get $p(S_2) \geq 2(2^{2k}) - q_{S_2} + 1$. Then by Theorem 8, we can move two pebbles to the vertex v.

Case (2) $v \in A_1$ *or* $v \in A_3$. Without loss of generality, assume that $v \in A_3$. Suppose that $p(A_3) \geq (\frac{2^{k+2}-(-1)^{k+2}}{3})$. Then we can move one pebble to v. This leaves at least $p(G) - (\frac{2^{k+2}-(-1)^{k+2}}{3})$ pebbles in G. We have to move an additional pebble to the vertex v by using these unused pebbles in G. Thus, we claim the following:

$Claim3 : p(G) - (\frac{2^{k+2}-(-1)^{k+2}}{3}) \geq f(G)$
It is enough to prove that $2^{3k-1} + 3k - 2 - q + 1 - (\frac{2^{k+2}-(-1)^{k+2}}{3}) \geq 0$.
$2^{3k-1} + 3k - 2 - q + 1 - (\frac{2^{k+2}-(-1)^{k+2}}{3}) \geq 2^{3k-1} - 3k - 2^{k+1}$
$= 2^{k+1}(2^{2k-2} - 1) - 3k$
$\geq 2^{k+1} - 3k$, since $k \geq 2$
> 0. Hence by claim 3, the number of pebbles retained in G is at least f(G). Then by Theorem 6, we can move an additional pebble to v. Therefore, assume that $p(A_3) < (\frac{2^{k+2}-(-1)^{k+2}}{3})$. Suppose $v \in V(A_3) - \{y, z, a_{3k-2}, b_{3k-2}\}$. By Claim: 3, we conclude that pebbles at S_1 is at least $f(G)$. Then by Theorem 7, we can move at least 2^{k-1} pebbles to the vertex a_{2k-1} or to the vertex b_{2k} by using exactly $f(G) - (\frac{2^{k+2}-(-1)^{k+2}}{3}) + 1$ pebbles. Therefore, we can transfer two pebbles from the vertex a_{2k-1} or from the vertex b_{2k} to the vertex v. Now, we take $v \in \{y, z, a_{3k-2}, b_{3k-2}\}$. Without loss of generality, assume that $v = y$. Let $C : a_1, a_2, ..., y$ and $D : b_2, b_3, ..., z, y$ and note that $f(C) = f(D) = 2^{3k-2}$. Let q_C and q_D be the number of occupied vertices in C and D respectively. Then we are done. Suppose not, we must have the following inequalities: $p(C) \leq 2(2^{3k-2}) - q_C$ and $p(D) \leq 2(2^{3k-2}) - q_D$. Without loss of generality, assume that $p(C) \geq p(D)$. Suppose $p(C) \geq 2^{3k-2}$. Then we can move one pebble to the vertex v, by using exactly 2^{3k-2} pebbles. This leaves at least $p(G) - 2^{3k-2}$ pebbles in G. We have to move an additional pebble to v. Thus, we claim the following:
$Claim4 : p(G) - 2^{3k-2} \geq f(G)$
We have, $p(G) - 2^{3k-2} = 3(2^{3k-2}) + 6k - 3 - (3(2k - 1) + 2)$
$\geq 2^{3k-1} + 2^{3k-2} - 2$
$\geq 2^{3k-1} + 3k - 2$, for $k \geq 2$. Thus by claim 4, we conclude that, the number of pebbles retained in G is at least $f(G)$. So we can move an additional pebble to v.

Now, we assume that $p(C) \leq 2^{3k-2} - 1$. Let $C' = C - y$ and $D' = D - y$ be two paths of length $3(k-1)$. Suppose any one of these paths satisfies the two-pebbling property. This completes the proof. Otherwise, we claim the following:
$Claim5 : p(G) - p(C') \geq f(G)$
Since, $p(C') \leq 2^{3k-2}$, this inequality follows from the Claim: 4. Therefore, assume that $p(C') \leq 2(2^{3k-3}) - q_C$ and $p(D') \leq 2(2^{3k-3}) - q_D$. Without loss of generality, assume that $q_C = q_D = 0$, then all the pebbles are in x and b_1. Clearly, at least 2^{3k-1} pebbles are in any one of the vertices in $\{x, b_1\}$. Take $p(x) \geq 2^{3k-1}$. Then we can transfer one pebble from the vertex x to the vertex v. We have to move an additional pebble to v. We claim the following:
$Claim6 : p(G) - 2^{3k-1} \geq f(G)$
We have, $p(G) - 2^{3k-1} = 2^{3k-1} - q + 1 - 2^{3k-1} + 6k - 4$
$= 2^{3k-2} + 3k - 2 + 3k - 3$
$> 2^{3k-2} + 3k - 2$, for $k \geq 2$. Hence by claim at least $f(G)$ pebbles are retained in G.

□

Theorem 10. *Let $G \cong ZZ_n$ be the zig-zag chain graph of n copies of odd cycles. Suppose that $p(G) \geq 2f(G) - q + 1$. Then we can move two pebbles to the vertex $v \in V(G)$.*

Proof. We proceed by induction on m. For $m = 1$, the result follows from Theorem 8 and Theorem 9. Let q_i be the number of occupied vertices in A_i, $1 \leq i \leq n$. Take $v \in A_p$, for $1 \leq p \leq n$. Now, the graph G can be partitioned into two subgraphs, say $S_1 \cong ZZ_p$ and $S_2 \cong ZZ_s$. Here, $S_1 \cap S_2 \cong A_p$. Clearly, $v \in S_1$. For convenience, we may assume that q_{S_i} be the number of occupied vertices in S_i, for $i = 1, 2$. We can write $q = q_{S_1} + q_{S_2} - q_p$. We consider the following possibilities:

Case (1) *n is even.* Take $G \cong ZZ_{2m}$. Let D be a distribution with at least $2f(G) - q + 1$ pebbles. Let $v \in A_p, 1 \leq p \leq 2m$. We have following three subcases:
Subcase $(1a)$: *p is odd and $1 < p < 2m$.* Clearly, $v \in S_1$ and also we know that $2m = p + s - 1$. Here, we can put $p = 2p' + 1$ and thus we have $s = 2s'$. Suppose that $p(S_1) \geq 2f(S_1) - q_{S_1} + 1$. Then we are done. Otherwise, we claim the following:
$Claim7 : p(G) - p(S_1) \geq 2f(S_2) - q_{S_2} + 1$
$p(G) - p(S_1) = 2(2^{(p'+s')(2k-1)+1} - 2^{(2p'+1)k-p'}) - 2(2p'k + k - p' - 1) - q_{S_2} + q_p$
$\geq 2(2^{s'(2k-1)+1}) + 2^{p'(2k-1)+1}(2^{s'(2k-1)} - 2^k) - 2(2p'k + k - p' - 1) - q_{S_2} + q_p$
$\geq 2(2^{s'(2k-1)+1}) + 2^{4k-1} - 2^{k+1} - 6k + 4 - q_{S_2} + q_p$, since $p', s' \geq 1$
$> 2(2^{s'(2k-1)+1}) - q_{S_2} + 1$. Hence by Claim 7, $p(S_2) \geq 2f(S_2) - q_{S_2} + 1$. Then by induction, we can move two pebbles to the vertex v.
Subcase $(1b)$: *p is even and $1 < p < 2m$.* Here, we can put $p = 2p'$ and thus we have $s = 2s' + 1$. Suppose the subgraph S_1 satisfies the two-pebbling property. Then we are done. Otherwise we claim the following:
$Claim8 : p(G) - p(S_1) \geq 2f(S_2) - q_{S_2} + 1$

We can see that , this inequality follows from the above Claim 7, by simply replacing the position of S_1 and S_2. Thus, we conclude that at least $2f(S_2) - q_{S_2} + 1$ unused pebbles are retained in S_2. Now, by induction, we put two pebbles on the vertex v.

Subcase $(1c)$: $p = 1$ *or* $p = 2m$. Let us take $v \in A_{2m}$ and also $p(A_{2m}) < (\frac{2^{k+2}-(-1)^{k+2}}{3})$. Without loss of generality, assume that $v = y$. Let $E : a_1, a_2, ..., y$ and $F : b_1, b_2, ..., y$ be two paths and recall that $f(E) = f(F) = 2^{m(2k-1)}$. Let q_E and q_F be the number of occupied vertices in E and F respectively. Suppose any one of these paths satisfies the two-pebbling property. Then we are done. Suppose not, we must have the following two inequalities: $\frac{p(x)}{2} + p(E) \geq 2(2^{m(2k-1)}) - q_E + 1$ or $\frac{p(x)}{2} + p(F) \geq 2(2^{m(2k-1)}) - q_F + 1$. Without loss of generality, assume that $\frac{p(x)}{2} + p(E) \geq 2(2^{m(2k-1)}) - q_E + 1$. Then we can move two pebbles to the vertex v. Otherwise, $p(x) + p(E) + p(F) < 2(2^{m(2k-1)+1}) - q + 1$, which contradicts the total number of pebbles distributed in G.

Case (2) *n is odd.* Let $G \equiv ZZ_{2m+1}$ with at least $2f(G) - q + 1$ pebbles distributed on its vertices. We consider the following three subcases:

Subcase $(2a)$: *p is odd and* $1 < p < 2m + 1$. We know that $2m + 1 = p + s - 1$. Here, we put $p = 2p' + 1$ and thus we have $s = 2s' + 1$. Suppose the subgraph S_1 satisfies the two-pebbling property. Then by induction we can move two pebbles to the vertex v. Suppose not, we claim the following:

$Claim 9 : p(G) - p(S_1) \geq 2f(S_2) - q_{S_2} + 1$

We have, $p(G) - p(S_1)$

$= 2(2^{2k(p'+s')+k-(p'+s')} - 2^{(2p'+1)k-p'}) + 2k(2s' + 1) - 2s' - k - q_{S_2} + q_p + 1$

$\geq 2(2^{(2s'+1)k-s'}(2^{2k-1}) - 2^{3k-1}) + 2k(2s' + 1) - 2s' - k - q_{S_2} + q_p + 1$, since $p' \geq 1$

$> 2f(S_2) - q_{S_2} + 1$, since $s' \geq 1$. Hence by Claim 9, we conclude that the subgraph S_2 satisfies the two pebbling property. Then we are done.

Subcase $(2b)$: *p is even and* $1 < p < 2m + 1$. We can take $p = 2p'$ and thus we have $s = 2s'$. Suppose that the subgraph S_1 satisfies the two-pebbling property. Then by induction, we can move two pebbles to the vertex v. Otherwise, we claim the following:

$Claim 10 : p(G) - p(S_1) \geq 2f(S_2) - q_{S_2} + 1$

We have, $p(G) - p(S_1) \geq 2(2^{s'(2k-1)+1}(2^{p'(2k-1)-k}) - 2^{p'(2k-1)+1}) - q_{S_2} + q_p + 1$

$\geq 2(2^{s'(2k-1)+1}) - q_{S_2} + 1 + 2^{2k}(2^{k-2} - 1)$, since $p', s' \geq 1$ and $k \geq 2$

$\geq 2f(S_2) - q_{S_2} + 1$. Then by Claim: 10, we may conclude that $p(S_2) \geq 2f(S_2) - q_{S_2} + 1$. By induction, we can move two pebbles to the vertex v.

Subcase $(2c)$: $p = 1$ *or* $p = 2m + 1$. Without loss of generality, assume that $v \in A_{2m+1}$. Suppose that $p(A_{2m+1}) \geq (\frac{2^{k+2}-(-1)^{k+2}}{3})$. Then we can move one pebble to the vertex v by Theorem 5. This leaves at least $f(G)$ pebbles in G. Then we can move an additional pebble to the vertex v. This completes the proof. Therefore, assume that $p(A_{2m+1}) < (\frac{2^{k+2}-(-1)^{k+2}}{3})$. Suppose $v \in V(A_{2m+1}) - \{a_{(2m+1)k-(m+1)}, b_{(2m+1)k-(m+1)}, z, y\}$. Since by our assumption, pebbles at $< G - A_{2m} >$ are at least $f(G) - (\frac{2^{k+2}-(-1)^{k+2}}{3})$. Then by Theorem 7, we can move 2^{k-1} pebbles to the vertex a_{2k-1}. So that we can reach our target with two pebbles. Now, let $v \in \{a_{(2m+1)k-(m+1)}, b_{(2m+1)k-(m+1)}, z, y\}$. Take $v = y$. We have two paths $X : a_1, a_2, ..., y$ and $Y : b_1, b_2, ..., y$ of length $(2m + 1)k - (m + 1)$. Let q_X and $q_Y$be number of occupied vertices in X and Y respectively. Suppose any one of these paths satisfies the two-pebbling property. Then we are done. otherwise, we can define two paths X' and Y' by removing the vertex y from the path X and the path Y respectively. Suppose $p(X') \geq 2f(X') - q_{X'} + 1$ or $p(Y') \geq 2f(Y') - q_{Y'} + 1$.

We may assume that $p(X') \geq 2f(X') - q_{X'} + 1$. Then we can move one pebble to the vertex y. This leaves at least $f(G)$ unused pebbles in G. So, we can move an additional pebble to the vertex y by Theorem 6. Therefore without loss of generality assume that $q_{X'} = q_{Y'} = 0$. Then all the pebbles are retained in the vertices x and b_1. Clearly at least $2^{(2m+1)k-m}$ pebbles are retained in any one of these vertices. We may take $p(x) \geq 2^{(2m+1)k-m}$. Then we can move one pebble to the vertex v. We have to move an additional pebble to the vertex v. Thus we claim the following:
$Claim 11 : p(G) - 2^{(2m+1)k-m} \geq f(G)$
$p(G) - 2^{(2m+1)k-m} = 2(2^{(2m+1)k-m} + (2m+1)k - (m+1)) - 2 + 1 - 2^{(2m+1)k-m}$
$\geq 2^{(2m+1)k-m} + (2m+1)k - (m+1) + (m+1)(k-1) + mk - 1$
$> 2^{(2m+1)k-m} + (2m+1)k - (m+1)$. Hence by Claim: 11, we can conclude that the graph G has at least $f(G)$ unused pebbles after using $2^{(2m+1)k-m}$ pebbles. Thus by Theorem 6, we can move an additional pebble to the vertex y. □

4 The $2t-$pebbling property

In this section, we show that the graph ZZ_n satisfies the $2t-$pebbling property.

Theorem 11. *The graph $G \cong ZZ_{2m}$ satisfies the $2t-$pebbling property.*

Proof. We proceed by induction on t. For $t = 1$, the result follows from Theorem 10. Assume that the result is true for all $t' < t$. Let us take $G \cong ZZ_{2m}$ and let D be a distribution with more that $2f_t(G) - q$ pebbles on the vertices of the graph G. We have to move at least $2t$ pebbles to any specifed vertex $v \in G$. Consider the following possibilities:

Case (1) $p(v) = 0$. We have to move $2t$ pebbles to the vertex v. We can write $2f_t(G) - q + 1 = 2(f_{t-1}(G) + f_t(G)) - q + 1 \geq 2f(G)$. Therefore by Theorem 6, we can move two pebbles to v by using exactly $2f(G)$ pebbles. Then the number of unused pebbles distributed on G is at least $2f_{t-1}(G) - q + 1$. By induction, we can move additional $2(t-1)$ pebbles to the vertex v.

Case (2) $p(v) = x$ *and* $1 \leq x \leq 2t - 1$. We have to move at least $2t - x$ pebbles to the vertex x. Therefore, we consider the following subcases:
Subcase $(2a)$: x *is even.* Let us take $x = 2x_1$, where x_1 is any non-negative integer. We have to move $2(t - x_1)$ pebbles to the vertex v. Without considering the $2x_1$ pebbles distributed on the vertex v, we must have at least $2f_{t-x_1}(G) - q + 1$ unused pebbles in G. This follows from the below claim:
$Claim 12 : p(G) - 2x_1 \geq 2f_{t-x_1}(G) - q + 1$
We have, $p(G) - 2x_1 = 2(f_{t-x_1}(G) + f_{x_1}(G)) - q + 1 - 2x_1$
$> 2f_{t-x_1}(G) - q + 1$. Therefore, by claim we can move additional $2t - x$ pebbles to the vertex v.
Subcase $(2b)$: x *is odd.* Using exactly $2^{m(2k-1)+1}$ pebbles, we can move an additional pebble to v. So, $p(v) = x + 1 = 2x_2$, where x_2 is any non-negative integer. Therefore, we have to move additional $2(t - x_2)$ pebbles to the vertex v. Now, we claim the following:
$Claim 13 : p(G) - x - 2^{m(2k-1)+1} \geq 2f_{t-x_2} - q + 1$
We have, $p(G) - x - 2^{m(2k-1)+1} = 2(f_{t-x_2} + f_{x_2}) - q + 1 - f(G) - 2x_2$
$= 2f_{t-x_2}(G) - q + 1 + 2f_{x_2}(G) - f(G) - 2x_2$

$\geq 2f_{t-x_2}(G) - q + 1$. Thus by Claim: 13, we conclude that we can move at least $2(t - x_2)$ additional pebbles to the vertex v. □

Theorem 12. *Let $G \cong ZZ_{2m+1}$ and let D be a distribution with at least $2f_t(G) - q + 1$ pebbles on the vertices of G. Suppose $v \in A_p, 1 < p < 2m + 1$ and $p(v) = 0$. Then we can move at least $2t$ pebbles to the vertex v.*

Proof. We proceed by induction on t. For $t = 1$, the result follows from Theorem 10. Assume that the result is true for all $t' < t$. Consider the distribution of at least $2f_t(G) - q + 1$ pebbles on the vertices of G. Let q_i be the number of occupied vertices in A_i respectively. We can split the graph G into two subgraphs, say $S_1 \cong ZZ_p$ and $S_2 \cong ZZ_s$, where $S_1 \cap S_2 \cong A_p$. For convenience, we can take q_{S_1} and q_{S_2} be the number of occupied vertices in S_1 and S_2 respectively. Therefore, we can write $q = q_{S_1} + q_{S_2} - q_p$. We consider the following two possibilities:

Case (1) *p is odd.* We know that $2m + 1 = p + s - 1$. Put $p = 2p' + 1$ and thus we have $s = 2s' + 1$. Suppose $p(S_1) \geq 2f_t(S_1) - q_{S_1} + 1$. Then by induction, we can move $2t$ pebbles to the vertex v. Otherwise, we claim the following:
$Claim 14 : p(G) - p(S_1) \geq 2f_t(S_2) - q_{S_2} + 1$
It is enough to prove that, $f_t(G) - f_t(S_1) \geq f_t(S_2)$
$f_t(G) - f_t(S_1) = t(2^{(2m+1)k-m} - 2^{(2p'+1)k-p'}) + 2s'k - s'$
$= t(2^{(2s'+1)k-s'}(2^{2k-1}) - 2^{3k-1}) + 2s'k - s'$, since $p' \geq 1$
$> t2^{(2s'+1)k-s'} + k + 2s'k - s' - 1$. Hence by claim, we conclude that $p(S_2) \geq 2f_t(S_2) - q_{S_2} + 1$. Then by induction, we can move at least $2t$ pebbles to the vertex v.

Case (2) *p is even.* We can write $p = 2p'$ and thus we have $s = 2s'$. Suppose $p(S_1) \geq 2f_t(S_1) - q_{S_1} + 1$. Then by induction we can move $2t$ pebbles to the vertex v. Otherwise, we claim the following:
$Claim 15 : p(G) - p(S_1) \geq 2f_t(S_2) - q_{S_2} + 1$. This claim follows from the Claim 14, since $f(ZZ_{2s'+1}) \geq f(ZZ_{2s'})$. Hence, we can move at least $2t$ pebbles to the vertex v. □

Theorem 13. *The graph $G \cong ZZ_{2m+1}$ satisfies the $2t-$pebbling property.*

Proof. We proceed by induction on t. For $t = 1$, the result follows from the Theorem 10. We assume that the result is true for $t' < t$. Consider the graph G with at least $2f_t(G) - q + 1$ pebbles. Clearly v belongs to A_p, for $1 \leq p \leq 2m + 1$. We have following two possibilities:

Case (1) $p(v) = 0$. Suppose $v \in A_p, 1 < p < 2m + 1$. Then by Theorem 12, we can move at least $2t$ pebbles to the vertex v. So, we assume that $v \in A_1$ or $v \in A_{2m+1}$. Without loss of generality, assume that $v \in A_{2m+1}$ and also take $v = y$. We have two paths $U : a_1, a_2, ..., y$ and $W : b_2, b_3, ..., y$ each of length $(2m + 1)k - (m + 1)$. Let q_U and q_W be the number of occupied vertices in U and W respectively. Suppose any one these paths satisfies the $2t-$pebbling property. Then we are done. Otherwise, we can get two subpaths U' and W', by removing the vertex y from the paths U and W, respectively. Assume that $p(U') \geq p(W')$. Suppose $p(U') \geq 2f_t(U') - q_U + 1$. Then we can move t pebbles to the vertex v. This leaves at least $f_t(G)$ pebbles in G. Therefore we can move additional one peble to the vertex v. So, assume that $p(W') \leq p(U') < 2t(2^{(2m+1)k-(m+2)})$. Without loss of generality, we may take $q_U = q_W = 0$. Then all the pebbles are distributed only on the vertices x and b_1. Clearly, any one of these vertices must have at least $t2^{(2m+1)k-m}$. So that we can move at least t pebbles to the vertex v. We have to move additional t pebbles to the vertex v. Then the number of pebbles retained in G is at least $t2^{(2m+1)k-(m+1)} + (2m+1)k - (m+1)$. By Theorem 10, we can move additional t pebbles to the vertex v.

Case (2) $p(v) = x, 1 < x < 2t-1$. We have to move $2t - x$ additional pebbles to the vertex v. We have the following two subcases:

Subcase $(2a)$: *x is even.* Take $x = 2x_1$, where x_1 is any non-negative integer. Thus we have to move at least $2(t - x_1)$ pebbles to the vertex v. Without considering the pebbles on the vertex v, the graph G has at least $2f_{t-x_1} - q + 1$ pebbles on its vertices. Then by induction, we can move additional $2(t - x_1)$ pebbles to the vertex v.

Subcase $(2b)$: *x is odd.* We have two paths $P : x, a_1, ..., y$ and $Q : b_1, b_2, ..., y$ each of length $(2m+1)k - m$. Clearly, any one of these paths has at least $2^{(2m+1)k-m}$ pebbles. Then we can move one pebble to the vertex v. So $p(v) = x + 1 = 2x_2$. Now, we have to move $2(t - x_2)$ additional pebbles to the vertex v. Thus, we claim the following:

$Claim 16 : p(G) - x - 2^{(2m+1)k-m} \geq 2f_{t-x_2}(G) - q + 1$

We have, $p(G) - x - 2^{(2m+1)k-m} \geq 2f_{t-x_2}(G) - q + 1 + 2f_{x_2} - 2x_2 - 2^{(2m+1)k-m}$ $> 2f_{t-x_2}(G) - q + 1$. Hence by claim 16, we conclude that, the number of pebbles retained in G is at least $2f_{t-x_2}(G) - q + 1$. By induction, we can move $2(t - x_2)$ additional pebbles to the vertex v.

□

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