Superior Eccentric Domination in Graphs

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Abstract

In this paper we define superior eccentric dominating set. A superior dominating set S of vertices of G is called a superior eccentric dominating set if every vertex of V(G) − S has some superior eccentric vertex in S. A superior eccentric dominating set of G of minimum cardinality is a minimum superior eccentric dominating set and its cardinality is called the superior eccentric domination number and is denoted by γ_{sed}(G). We show that for each pair of positive integers m, n there is a connected graph G with domination number γ(G) = m and superior eccentric domination number γ_{sed}(G) = n.

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1 Introduction

Let G be a finite, simple, undirected (a,b) graph with vertex set V(G) and edge set E(G), |V(G)| = a, |E(G)| = b. For graph theoretic terminology refer Harary [3], Buckley and Harary [1].


A set D ⊆ V is said to be a dominating set in G, if every vertex in V − D is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number and is denoted by γ(G). Let G be a connected graph and v be a vertex of G. The eccentricity e(v) of v is the distance to a vertex farthest from v. Thus, e(v) = max{d(u,v) : u ∈ V}. A set D ⊆ V(G) is an eccentric dominating set if
D is a dominating set of G and for every v ∈ V−D, there exist at least one eccentric vertex of v in D. The minimum cardinality of an eccentric dominating set is called the eccentric domination number and is denoted by $\gamma_{ed}(G)$. For two vertices u and v in a graph G, the distance from u to v is denoted by d(u, v) and defined as the length of a distance u−v path in graph G.

For distinct vertices u and v of a non-trivial connected graph G, we let $D_{u,v} = N(u) \cup N(v)$. We define a $D_{u,v}$ walk as a u−v walk in G that contains every vertex of $D_{u,v}$. The superior distance $d_{sd}(u,v)$ from u to v is the length of a shortest $D_{u,v}$ walk. For each vertex $u \in V(G)$, define $d_{sd}(u) = \min\{d_{sd}(u, v) : v \in V(G) - \{u\}\}$. A vertex v ($\neq u$) is called a superior neighbor of u if $d_{sd}(u, v) = d_{sd}(u)$. A vertex u is said to superior dominate a vertex v if v is a superior neighbor of u. A set S of vertices of G is called a superior dominating set if every vertex of V(G) − S has some superior eccentric vertex in S. A superior dominating set of G of minimum cardinality is a minimum superior dominating set and its cardinality is called the superior domination number of G and is denoted by $\gamma_{sd}(G)$. We define the superior eccentricity of v as $e_{sd}(v) = \max\{d_{sd}(u,v) : u \in V(G)\}$. A vertex v of a graph G is said to be a superior eccentric vertex of a vertex u if $d_{sd}(u,v) = e_{sd}(u)$. A vertex u is superior eccentric vertex of G if it is a superior eccentric vertex of some vertex v.

2 Superior Eccentric Domination in Graphs

Definition A superior dominating set S of vertices of G is called a superior eccentric dominating set if every vertex of V(G) − S has some superior eccentric vertex in S.

**Theorem 1.**
1. $\gamma_{sed}(K_{1,n}) = 2$.
2. $\gamma_{sed}(K_n) = 1$.

**Proof.** (1) $G = K_{1,n}$. Let S = \{v, v_n\}. The central vertex v is superior dominates all vertices in V− S and v is an superior eccentric point of vertices in V−D that is $d_{sd}(v,v_i) = 2n-1$, $d_{sd}(v_i, v_j) = 2$. Superior neighbor of $v_i$ is $v_j$, $j \neq i$, superior eccentric vertices of $v_j$ is $v_i$. Hence $\gamma_{sed}(K_{1,n}) = 2$.

(2) When G = $K_n$, radius = diameter = 1. Hence any vertex $u \in V(G)$ superior dominates other vertices and is also a superior eccentric point of other vertices. Hence $\gamma_{sed}(K_n) = 1$.

**Theorem 2.**
(i) $\gamma_{sed}(P_4) = 2$. (ii) $\gamma_{sed}(P_5) = 3$. (iii) $\gamma_{sed}(P_6) = 4$. (iv) $\gamma_{sed}(P_7) = 4$. (v) $\gamma_{sed}(P_8) = 4$. (vi) $\gamma_{sed}(P_9) = 5$.

**Proof.** **Proof of (i)** Let $v_1, v_2, v_3, v_4$ represent the vertices of P_4. Then $N_{sd}(v_1) = \{v_3\}, N_{sd}(v_2) = \{v_1, v_3\}, N_{sd}(v_3) = \{v_1, v_4\}, N_{sd}(v_4) = \{v_4\}$ and superior eccentric vertex of $v_1$ is $v_3$, superior eccentric vertex of $v_2$ is $v_1, v_3$, superior eccentric vertex of $v_3$ is $v_2$, superior eccentric vertex of $v_4$ is $v_2$.

Therefore $S = \{v_2, v_3\}$ is the only superior eccentric dominating set. Therefore $\gamma_{sed}(P_4) = 2$.

**Proof of (ii)** Let $v_1, v_2, v_3, v_4, v_5$ represent the vertices of P_5. Then $N_{sd}(v_1) = v_2, N_{sd}(v_2) = v_1, N_{sd}(v_3) = \{v_1, v_5\}, N_{sd}(v_4) = \{v_5\}, N_{sd}(v_5) = \{v_4\}$ and superior
eccentric vertex of is \( v_1 \) is \( v_1 \), superior eccentric vertex of \( v_2 \) is \( v_4 \), superior eccentric vertex of \( v_3 \) is \( v_2 \), superior eccentric vertex of \( v_4 \) is \( v_2 \), superior eccentric vertex of \( v_5 \) is \( v_2 \).

Therefore \( S = \{v_2, v_3, v_4\} \) is the only superior eccentric dominating set. Therefore \( \gamma_{sed}(P_5) = 3 \)

**Proof of (iii)** Let \( v_1, v_2, v_3, v_4, v_5, v_6 \) represent the vertices of \( P_6 \). Then \( N_D(v_1) = v_2, N_D(v_2) = v_1, N_D(v_3) = \{v_1\}, N_D(v_4) = \{v_6\}, N_D(v_5) = \{v_6\}, N_D(v_6) = \{v_3\} \) and superior eccentric vertex of is \( v_1 \) is \( v_5 \), superior eccentric vertex of \( v_2 \) is \( v_5 \), superior eccentric vertex of \( v_3 \) is \( v_5 \), superior eccentric vertex of \( v_4 \) is \( v_2 \), superior eccentric vertex of \( v_5 \) is \( v_2 \), superior eccentric vertex of \( v_6 \) is \( v_2 \).

Therefore \( S = \{v_1, v_2, v_3, v_6, v_6\} \) is the minimum superior eccentric dominating set. Therefore \( \gamma_{sed}(P_6) = 4 \).

**Proof of (iv)** Let \( v_1, v_2, v_3, v_4, v_5, v_6, v_7 \) represent the vertices of \( P_7 \). Then \( N_D(v_1) = \{v_2\}, N_D(v_2) = \{v_1\}, N_D(v_3) = \{v_1\}, N_D(v_4) = \{v_5\}, N_D(v_5) = \{v_7\}, N_D(v_6) = \{v_6\} \) and superior eccentric vertex of \( v_1 \) is \( v_6 \), superior eccentric vertex of \( v_2 \) is \( v_6 \), superior eccentric vertex of \( v_3 \) is \( v_6 \), superior eccentric vertex of \( v_4 \) is \( v_6 \), superior eccentric vertex of \( v_5 \) is \( v_2 \), superior eccentric vertex of \( v_6 \) is \( v_2 \), superior eccentric vertex of \( v_7 \) is \( v_2 \).

Therefore \( S = \{v_1, v_2, v_6, v_7\} \) is the minimum superior eccentric dominating set. Therefore \( \gamma_{sed}(P_7) = 4 \).

**Proof of (v)** Let \( v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \) represent the vertices of \( P_8 \). Then \( N_D(v_1) = \{v_2\}, N_D(v_2) = \{v_1\}, N_D(v_3) = \{v_1\}, N_D(v_4) = \{v_5\}, N_D(v_5) = \{v_7\}, N_D(v_6) = \{v_8\}, N_D(v_7) = \{v_8\}, N_D(v_8) = \{v_7\} \) and superior eccentric vertex of \( v_1 \) is \( v_7 \), superior eccentric vertex of \( v_2 \) is \( v_7 \), superior eccentric vertex of \( v_3 \) is \( v_7 \), superior eccentric vertex of \( v_4 \) is \( v_7 \), superior eccentric vertex of \( v_5 \) is \( v_7 \), superior eccentric vertex of \( v_6 \) is \( v_2 \), superior eccentric vertex of \( v_7 \) is \( v_2 \), superior eccentric vertex of \( v_8 \) is \( v_2 \).

Therefore \( S = \{v_1, v_2, v_7, v_8\} \) is the minimum superior eccentric dominating set. Therefore \( \gamma_{sed}(P_8) = 4 \).

**Proof of (vi)** Let \( v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9 \) represent the vertices of \( P_9 \). Then \( N_D(v_1) = \{v_2\}, N_D(v_2) = \{v_1\}, N_D(v_3) = \{v_1\}, N_D(v_4) = \{v_5\}, N_D(v_5) = \{v_7\}, N_D(v_6) = \{v_8\}, N_D(v_7) = \{v_8\}, N_D(v_8) = \{v_9\}, N_D(v_9) = \{v_8\} \) and superior eccentric vertex of \( v_1 \) is \( v_8 \), superior eccentric vertex of \( v_2 \) is \( v_8 \), superior eccentric vertex of \( v_3 \) is \( v_8 \), superior eccentric vertex of \( v_4 \) is \( v_8 \), superior eccentric vertex of \( v_5 \) is \( v_8 \), superior eccentric vertex of \( v_6 \) is \( v_2 \), superior eccentric vertex of \( v_7 \) is \( v_2 \), superior eccentric vertex of \( v_8 \) is \( v_2 \), superior eccentric vertex of \( v_9 \) is \( v_2 \).

Therefore \( S = \{v_1, v_3, v_5, v_8, v_9\} \) is the minimum superior eccentric dominating set. Therefore \( \gamma_{sed}(P_9) = 5 \).

In general, proof:

**Case (i)** \( \gamma_{sed}(P_n) = \frac{n-3}{2}+4 \). Where \( n \) is even \( n \geq 10 \) and \( n = 2k \).

1) \( v_{2k−1} \) is the eccentric vertex of \( v_1, v_2, v_3, . . . , v_k \).
2) \( v_{2k} \) is the eccentric vertex of \( v_{k+1}, . . . , v_{2k} \).
3) \( v_1 \) is the neighbor of \( v_2, v_3 \).
4) \( v_{2k} \) is the neighbor of \( v_{2k−1} \) and \( v_{2k−2} \).
5) \( v_4 \) is the neighbor of \( v_{i+1} \) when \( 4 \leq i \leq 2k-2 \).
γ_{sed}(p_{n}) = \left\lfloor \frac{n-8}{3} \right\rfloor + 4 \text{ where } n \geq 10 \text{ and } n = 2k.

**Case (ii)** When n is odd, let n = 2k+1.
1) v_{2k} is the eccentric vertex of v_1, v_2, \ldots, v_{k+1}.
2) v_2 is the eccentric vertex of v_{k+1}, v_{k+2}, \ldots, v_{2k}, v_{2k+1}.
3) v_1 is the neighborhood of v_2, v_3.
4) v_{2k+1} is the neighbor of v_2, v_{2k-1}.
5) v_i is the neighbor of v_{i+1}, when 4 \leq i \leq 2k-2.

\[ \gamma_{sed}(C_n) = \begin{cases} 
\left\lfloor \frac{n}{3} \right\rfloor & \text{if } n = 3k \\
\left\lceil \frac{n}{3} \right\rceil & \text{if } n = 3k + 1 \\
\left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n = 3k + 2
\end{cases} \]

**Theorem 3.** Let G = C_n. when n is even \( \gamma_{sed}(c_n) = \frac{n}{2} \). When n is odd \( \gamma_{sed}(c_n) \).

**Proof.** Let v_0, x_1, v_1, x_2, v_2, x_3, v_3, x_4, \ldots, v_{n-1}, x_n, v_n represent the cycle C_n.

**In C_4 graph,** \( N_D(v_1) = \{v_2, v_4\} \), \( N_D(v_2) = \{v_1, v_3\} \), \( N_D(v_3) = \{v_2, v_4\} \), \( N_D(v_4) = \{v_1, v_3\} \). Superior eccentric vertex of v_1 is v_3, superior eccentric vertex of v_2 is v_4, superior eccentric vertex of v_3 is v_1, superior eccentric vertex of v_4 is v_2.

Therefore \( S = \{v_2, v_4\} \) is a superior eccentric dominating set and \( \gamma_{sed} = 2 \).

**In C_5 graph,** \( N_D(v_1) = \{v_2, v_5\} \), \( N_D(v_2) = \{v_1, v_3\} \), \( N_D(v_3) = \{v_2, v_4\} \), \( N_D(v_4) = \{v_3, v_5\} \), \( N_D(v_5) = \{v_1, v_4\} \). Superior eccentric vertex of v_1 is v_5, superior eccentric vertex of v_2 is v_4, superior eccentric vertex of v_3 is v_1, superior eccentric vertex of v_4 is v_1, superior eccentric vertex of v_5 is v_2.

Therefore \( S = \{v_1, v_5\} \) is a superior eccentric dominating set, \( \gamma_{sed} = 3 \).

In general,

When n is even that is n = 2k, \( d_D(v_i, v_{i+k}) = 2+k+2 = k+4 \), \( v_{i+k} \) is the superior eccentric vertex of \( v_i \). \( v_{i+1} \) and \( v_{i-1} \) are superior adjacent vertices of \( v_i \). Hence \( \gamma_{sed}(C_n) = n/2 \), if n is even.

When n is odd, each vertex of C_n has exactly two eccentric vertices. If n = 2k+1, \( v_i \in v(G) \) has \( v_{i+k}, v_{i+k+1} \) are eccentric vertices.

**Case (i)** \( n = 3m \), n is odd, m is odd

\[ n = 3m = 2k+1 \Rightarrow 2k \text{ is even and } 2k = 3m-1 \]
\[ \Rightarrow 2k = 3(m-1)+2 \]
\[ \Rightarrow k = (3(m-1)+2)/2 \]
\[ k = 3l+1 \text{ since } m-1 \text{ is even, where } l = m-1. \]

**Case (i) :** Consider \( S = \{v_1, v_4, v_7, \ldots, v_k, v_{k+3}, \ldots, v_{2k-1}\} \). S is an superior eccentric dominating set. S is a \( \gamma_{sed} \) superior dominating set of \( C_n \) and \( |S| = n/3 = m \). Hence \( \gamma_{sed}(C_n) = n/3 \).

**Case (ii) :** \( n = 3m+1 \), n is odd, m is even
Consider \( S = \{v_1, v_4, \ldots, v_{k+1}, v_{k+3}, v_{k+6}, \ldots, v_{2k-1}\} \). S is an superior eccentric dominating set and \( |S| = n/3 = \gamma(C_n) \). Hence \( \gamma_{sed}(C_n) = n/3 = m+1 \).

**Case (iii)** \( n = 3m+2, 3m \text{ is odd} \)
Consider $S = \{v_1, v_4, \ldots, v_{k-1}, v_k, v_{k+3}, \ldots, v_{2k+1}\}$. $S$ is an superior eccentric dominating set with $n/3 + 1$ vertices and $\gamma_{sed}(C_n) = n/3 + 1$.

**Lemma** For given integers $m$ and $n$ such that $m < n < 2m$ there exists a connected graph $G$ such that $\gamma(G) = m$, $\gamma_{sed}(G) = n$.

**Proof.** $m = 1$, $n \geq 2$.

Let $v_1, v_2$ and $v_3$ be the vertices of $K_3$. Construct a graph $G$ from the graph $K_3$ and $(n-1)$ copies of $P_3$ as follows:

Let $u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23}, \ldots, u_{(n-1)1}, u_{(n-1)2}, u_{(n-1)3}$ be the vertices of $(n-1)$ copies of $P_3$ where $u_{12}, u_{22}, u_{32}, \ldots, u_{(n-1)2}$ are the vertices of degree 2 in $P_3$. Let the resulting graph be $G$.

Now $d_D(v_1, v_2) = d_D(v_1, v_3) = 4(n-1)+2$

$d_D(v_1, u_{i1}) = 4(n-1)+2$ for all $i = 1, 2, 3, \ldots, n-1$

$d_D(v_1, u_{i3}) = 4(n-1)+2$ for all $i = 1, 2, 3, \ldots, n-1$

$d_D(v_1, u_{i2}) = 4(n-1)+3$ for all $i = 1, 2, 3, \ldots, n-1$.

$V_1$ superior eccentric dominates $v_1, v_2, u_{i1}, u_{i3}$ for all $i = 1, 2, 3, \ldots$, $n-1$. $u_{i2}$ for all $i = 1, 2, 3, \ldots, n-1$ are not superior eccentric dominated by any vertex of $G$. $v_2$ superior dominates $v_3$ and converse. Thus the set $S = \{v_1\} \cup \{u_{i2} / i = 1, 2, \ldots, n-1\}$ is a minimum superior eccentric dominating set of $G$ and hence $\gamma_{sed}(G) = n-1+1 = n$. Also $\gamma(G) = 1$ Construct a graph $G_1$ from the complete graph $K_m$ by merging a vertex of $K_3$ in each vertex $v_1, v_2, v_3, \ldots, v_m$ of $K_m$. Let $u_{11}, u_{12}, u_{21}, u_{22}, \ldots, u_{m1}, u_{m2}$ be the vertices of degree 2 in the graph $G_1$ where $u_{11}, u_{12}, u_{21}, u_{22}, \ldots, u_{m1}, u_{m2} \in E(G)$. Now subdivide $(n-m)$ edges $u_{11}, u_{12}, u_{21}, u_{22}, \ldots, u_{(n-m)1}, u_{(n-m)2}$ and let the new vertices be $w_1, w_2, \ldots, w_{(n-m)}$. Join these vertices with $v_1, v_2, \ldots, v_{n-m}$ respectively and let the resulting graph be $G$. $w_1, w_2, w_3, \ldots, w_{n-m}$ are not superior dominated by any vertex. $v_1$ superior eccentric dominates $u_{i1}, u_{i2}$ for each $i = 1, 2, 3, \ldots, a$. Thus the set $S = \{v_1, v_2, v_3, \ldots, v_m\} \cup \{w_1, w_2, \ldots, w_{n-m}\}$ forms a superior eccentric dominating set of $G$ and hence $\gamma_{sed}(G) = n$. Also $\gamma(G) = m$. $\square$
Lemma For any positive integer n, there exists a connected graph G such that \( \gamma(G) = \gamma_{sed}(G) = n \).

Proof. When n = 1, \( \gamma(K_2), \gamma_{sed}(K_2) = 1 \) for a given integer n \( \geq 2 \), consider the path \( P_n \). Let the vertices of \( P_n \) be \( v_1, v_2, v_3, \ldots, v_n \) and let \( e_i = v_i v_{i+1} \) be the edges of \( P_n \) for \( 1 \leq i \leq n-1 \). Let G be an arbitrary super subdivision of \( P_n \) for \( 1 \leq i \leq n-1 \), each \( e_i \) of \( P_n \) is replaced by a complete bipartite graph \( K_{2,m_i} \) where \( m_i \geq 2 \) is an integer (\( m_i \) may vary for each edge arbitrarily). Observe that G has \( 2(m_1 + m_2 + \ldots + m_n) \) edges.

The vertex set and the edge set of G are given below:
\[
V(H) = \{ v_1, u_{11}, u_{12}, \ldots, u_{1m_1}, v_2, u_{21}, u_{22}, \ldots, u_{2m_2}, \ldots, v_n, u_{(n-1)1}, u_{(n-1)2}, \ldots, u_{(n-1)m_{n-1}}, v_n \}
\]
where \( u_{ij} \) are the vertices which subdivide the edge \( v_i v_{i+1} \). \( u_{ij} \) are the vertices which subdivide the edge \( v_2 v_3 \) and so on. \( E(H) = \{ v_i u_{ij} : \text{for each } j = 1, 2, 3, \ldots, m_1 \} \cup \{ u_{ij} v_2 : \text{for each } j = 1, 2, 3, \ldots, m_1 \} \cup \{ v_{2} u_{ij} : \text{for each } j = 1, 2, 3, \ldots, m_2 \} \cup \{ v_{3} u_{ij} : \text{for each } j = 1, 2, 3, \ldots, m_2 \} \cup \ldots \cup \{ v_{n-1} u_{(n-1)j} : \text{for each } j = 1, 2, 3, \ldots, m_{n-1} \} \cup \{ u_{(n-1)j} v_n : \text{for each } j = 1, 2, 3, \ldots, m_{n-1} \} \). From this construction it is easy to verify that \( S = \{ v_1, v_2, v_3, \ldots, v_n \} \) is a minimal superior eccentric dominating set with minimum cardinality. Also S is a minimal superior dominating set and S is a minimal dominating set. Hence \( \gamma(G) = \gamma_{sed}(G) = n \).

References

