Abstract

The aim of this paper is to introduce a new class of generalized functions namely, $(1,2)^* - \alpha^*$-continuous functions and $(1,2)^* - \alpha^*$-irresolute functions and study its bitopological properties. Further, we investigate its relationship with other existing functions.

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1 Introduction

Levine [7] introduced generalized closed sets. Lellis Thivagar and Ravi [5] introduced $(1,2)^*$-bitopological spaces. Veronica Vijayan and Priya [10] defined and investigated the properties of $\alpha^*$ closed sets in topological space. Balachandran et al [3] studied about generalized continuity. Lellis Thivagar [9] defined $(1,2)^*$-semigeneralized continuous functions in bitopological spaces. Arockiarani and Mohana [2] deliberated $(1,2)^* - \pi g\alpha$-continuous functions in bitopological spaces. The purpose of this paper is to study about new class of functions, namely, $(1,2)^* - \alpha^*$-continuous functions and $(1,2)^* - \alpha^*$-irresolute functions in bitopological space. Finally, some of the basic properties of $(1,2)^* - \alpha^*$-continuous functions and $(1,2)^* - \alpha^*$-irresolute functions are investigated.
2 Preliminaries

"Throughout this paper, X and Y denote the bitopological spaces \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) respectively, on which no separation axioms are assumed."

**Definition 1.** [5] "A subset S of a bitopological space X is said to be \(\tau_{1,2}\)-open if \(S = A \cup B\) where \(\tau_1 \in A\) and \(\tau_2 \in B\). A subset S of X is said to be (i) \(\tau_{1,2}\)-closed if the complement of S is \(\tau_{1,2}\)-open. (ii) \(\tau_{1,2}\)-clopen if S is both \(\tau_{1,2}\)-open and \(\tau_{1,2}\)-closed."

**Definition 2.** [5] "Let S be a subset of the bitopological space X. Then the \(\tau_{1,2}\)-interior of S denoted by \(\tau_{1,2}\)-int(S) is defined by \(\bigcup\{G: G \subseteq S\) and G is \(\tau_{1,2}\)-open\} and the \(\tau_{1,2}\)-closure of S denoted by \(\tau_{1,2}\)-cl(S) is defined by \(\bigcap\{F: S \subseteq F\) and F is \(\tau_{1,2}\)-closed\}."

**Definition 3.** "A subset A of a bitopological space X is said to be

1. \((1,2)^*\) - regular open [5] if \(A = \tau_{1,2} - \text{int}(\tau_{1,2} - \text{cl}(A))\).
2. \((1,2)^*\) - semiopen [5] if \(A \subseteq \tau_{1,2} - \text{cl}(\tau_{1,2} - \text{int}(A))\).
3. \((1,2)^*\) - \(\alpha\)-open [5] if \(A \subseteq \tau_{1,2} - \text{int}(\tau_{1,2} - \text{cl}(\tau_{1,2} - \text{int}(A)))\).
4. \((1,2)^*\) - generalized \(\alpha\)-closed (briefly \((1,2)^*\)-g\(\alpha\)-closed) set[5] if \((1,2)^*\)-\(\alpha\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and U is \((1,2)^*\)-\(\alpha\)-open set in X
5. \((1,2)^*\) - generalized semi closed (briefly \((1,2)^*\)-gs-closed) set[5] if \((1,2)^*\)-scl(A) \subseteq U whenever \(A \subseteq U\) and U is \(\tau_{1,2}\)-open set in X
6. \((1,2)^*\) - \(\pi\) generalized closed (briefly \((1,2)^*\)-\(\pi\)g-closed) [8] if \(\tau_{1,2} - \text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and U is \(\tau_{1,2}\)-\(\pi\)-open set in X.
7. \((1,2)^*\) - \(\pi\) generalized \(\alpha\)-closed set[1] if \((1,2)^*\)-\(\alpha\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and U is \(\tau_{1,2}\)-\(\pi\)-open set in X.
8. \((1,2)^*\) - \(\alpha^*\)-closed (briefly \((1,2)^*\)-\(\alpha^*\)-closed) set[4] if \((1,2)^*\)-cl(A) \subseteq U whenever \(A \subseteq U\) and U is \((1,2)^*\)-\(\alpha\)-open set in X”

**Definition 4.** "A function \(f: X \rightarrow Y\) is called

1. \((1,2)^*\) - continuous [6] if the inverse image of every \(\sigma_{1,2}\)-closed set of Y is \(\tau_{1,2}\)-closed set in X.
2. \((1,2)^*\) - \(\alpha\)-continuous [6] if the inverse image of every \(\sigma_{1,2}\)-closed set of Y is \((1,2)^*\)-\(\alpha\)-closed set in X.
3. \((1,2)^*\) - \(\pi g\)-continuous [8] if the inverse image of every \(\sigma_{1,2}\)-closed set of Y is \((1,2)^*\)-\(\pi g\)-closed set in X.
4. \((1,2)^*\) - \(\pi g\alpha\)-continuous [2] if the inverse image of every \(\sigma_{1,2}\)-closed set of Y is \((1,2)^*\)-\(\pi g\alpha\)-closed in X.
5. 

(1, 2)*- go-continuous [5] if the inverse image of every \( \sigma_{1,2} \)-closed set of \( Y \) is 
(1, 2)*- go-closed in \( X \).

6. (1, 2)*- gs-continuous [5] if the inverse image of every \( \sigma_{1,2} \)-closed set of \( Y \) is 
(1, 2)*- gs-closed in \( X \).”

3 (1, 2)* - \( \alpha^* \) - Continuous Functions

Definition 5. A function \( f: X \rightarrow Y \) is called (1, 2)* - \( \alpha^* \) - continuous functions
if the inverse image of every \( \sigma_{1,2} \)-closed in \( Y \) is (1, 2)* - \( \alpha^* \) - closed in \( X \).

Example 3.1. Let \( X = Y = \{a, b, c\} \) with topologies \( \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}\}, \sigma_1 = \{\emptyset, Y, \{a, b\}\}, \sigma_2 = \{\emptyset, Y, \{c\}\} \). Let \( f: X \rightarrow Y \) be an identity function. Clearly, \( f \) is (1, 2)* - \( \alpha^* \) - continuous function.

Definition 3.3 A function \( f: X \rightarrow Y \) is called (1, 2)* - \( \alpha^* \) - irresolute functions if the
inverse image of (1, 2)* - \( \alpha^* \) - closed in \( Y \) is (1, 2)* - \( \alpha^* \) - closed in \( X \).

Example 3.4 Let \( X = Y = \{a, b, c, d\} \) with topologies \( \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}\}, \sigma_1 = \{\emptyset, Y, \{a, b, d\}\}, \sigma_2 = \{\emptyset, Y, \{c\}\} \). Let \( f: X \rightarrow Y \) be an identity function. Clearly, \( f \) is
(1, 2)* - \( \alpha^* \) - irresolute function.

Theorem 3.5 Every (1, 2)* - continuous function is (1, 2)* - \( \alpha^* \) - continuous function.

Proof: Obvious.

Remarks 3.6: Example 3.7 shows that the converse of the above theorem need not be true in general.

Example 3.7 Let \( X = Y = \{a, b, c\} \) with topologies \( \tau_1 = \{\emptyset, X, \{b\}\}, \tau_2 = \{\emptyset, X, \{a, c\}\}, \sigma_1 = \{\emptyset, Y, \{a, b\}\}, \sigma_2 = \{\emptyset, Y, \{a, c\}\} \). Let \( f: X \rightarrow Y \) be an identity function. Then \( f \) is (1, 2)* - \( \alpha^* \) - continuous but not (1, 2)* - continuous since \( f^{-1}(\{b\}) = \{b\} \) is not \( \tau_{1,2} \)-closed in \( X \), for \( \sigma_{1,2} \) closed set \( \{b\} \) in \( Y \).

Theorem 3.8 Every (1, 2)* - \( \alpha^* \) - continuous function is (1, 2)* - \( \alpha \) - continuous function.

Proof: The proof is clear.

Remarks 3.9: Example 3.10 shows that the converse of the above theorem need not be true in general.

Example 3.10 Let \( X = Y = \{a, b, c\} \) with topologies \( \tau_1 = \{\emptyset, X, \{a\}\}, \tau_2 = \{\emptyset, X, \{a, c\}\}, \sigma_1 = \{\emptyset, Y, \{b\}\}, \sigma_2 = \{\emptyset, Y, \{a, b\}\} \). Define a function \( f: X \rightarrow Y \) by \( f(a) = b, f(b) = a, f(c) = c \). Then \( f \) is (1, 2)* - \( \alpha \) - continuous but not (1, 2)* - \( \alpha^* \) - continuous.
since $f^{-1}(\{c\})=\{c\}$ is not $(1,2)^* - \alpha^* -$ closed set in $X$, for $\sigma_{1,2}$ closed set $\{c\}$ in $Y$.

**Theorem 3.11** Every $(1,2)^* - \alpha^* -$ continuous function is $(1,2)^* - \pi g^* -$ continuous function.

**Proof:** The proof is immediate.

**Remarks 3.12:** Example 3.13 shows that the converse of Theorem 3.11 need not true in general.

**Example 3.13** From example 3.10, $f$ is $(1,2)^* - \pi g^* -$ continuous but not $(1,2)^* - \alpha^* -$ continuous because $f^{-1}(\{c\})=\{c\}$ is not $(1,2)^* - \alpha^* -$ closed set in $X$, where $\{c\}$ is $\sigma_{1,2}$ closed set in $Y$.

**Theorem 3.14** Every $(1,2)^* - \alpha^* -$ continuous function is $(1,2)^* - \alpha g^* -$ continuous function.

**Proof:** The proof follows from the definition.

**Remarks 3.15:** Example 3.16 shows that the converse of the above theorem need not true in general.

**Example 3.16** From example 3.10, $f$ is $(1,2)^* - \alpha g^* -$ continuous but not $(1,2)^* - \alpha^* -$ continuous since $f^{-1}(\{c\})=\{c\}$ is not $(1,2)^* - \alpha^* -$ closed set in $X$, for $\{c\}$ is $\sigma_{1,2}$ closed set in $Y$.

**Theorem 3.17** Every $(1,2)^* - \alpha^* -$ continuous function is $(1,2)^* - g\alpha^* -$ continuous function.

**Proof:** The proof is obvious.

**Remarks 3.18:** Example 3.19 shows that the converse of the above theorem need not true in general.

**Example 3.19** Let $X = Y = \{a, b, c\}$ with topologies $\tau_1 = \{\varnothing, X, \{a\}\}$, $\sigma_1 = \{\varnothing, Y, \{a, c\}\}$, $\sigma_2 = \{\varnothing, Y, \{a\}, \{a, b\}\}$. Let $f : X \to Y$ be an identity function. Then $f$ is $(1,2)^* - g\alpha^* -$ continuous but not $(1,2)^* - \alpha^* -$ continuous , for $\sigma_{1,2}$ closed set $\{c\}$ in $Y$, since $f^{-1}(\{c\})=\{c\}$ is not $(1,2)^* - \alpha^* -$ closed set in $X$.

**Theorem 3.20** Every $(1,2)^* - \alpha^* -$ continuous function is $(1,2)^* - \pi g\alpha^* -$ continuous function.

**Proof:** The proof is immediate.

**Remarks 3.21:** Example 3.22 shows that the converse of the above theorem need not true in general.

**Example 3.22** Let $X = Y = \{a, b, c, d\}$ with topologies $\tau_1 = \{\varnothing, X, \{a\}\}$, $\tau_2 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$, $\sigma_1 = \{\varnothing, Y, \{a, b\}\}$, $\sigma_2 = \{\varnothing, Y, \{c\}\}$. Define $f : X \to Y$ be an identity function. Then $f$ is $(1,2)^* - g\alpha^* -$ continuous but not $(1,2)^* - \alpha^* -$ continuous , since for $\sigma_{1,2}$
closed set \{c, d\}, \{a, b, d\} in Y, \(f^{-1}(\{c, d\}, \{a, b, d\}) = \{c, d\}, \{a, b, d\}\) are not \((1, 2)^* - \alpha^* -\) closed set in X.

**Theorem 3.23** Every \((1, 2)^* - \alpha^* -\) continuous function is \((1, 2)^* - gs -\) continuous function.

**Proof:** The proof is obvious.

**Remarks 3.24:** Example 3.25 shows that the converse of the above theorem need not true in general.

**Example 3.25** From example 3.19, \(f\) is \((1, 2)^* - gs -\) continuous but not \((1, 2)^* - \alpha^* -\) continuous since \(f^{-1}(\{c\}) = \{c\}\) is not \((1, 2)^* - \alpha^* -\) closed set in X, for \(\{c\}\) is \(\sigma_{1,2}\) closed set in Y.

**Remark 3.26** The following diagram summarizes the above discussion.

1. \((1, 2)^* - \alpha^* -\) continuous 2. \((1, 2)^* -\) continuous 3. \((1, 2)^* - \pi g -\) continuous
4. \((1, 2)^* - \alpha g -\) continuous 5. \((1, 2)^* - g \alpha -\) continuous 6. \((1, 2)^* - \pi g \alpha -\) continuous
7. \((1, 2)^* - gs -\) continuous

**Theorem 3.27** A function \(f: X \rightarrow Y\) is \((1, 2)^* - \alpha^* -\) continuous iff \(f^{-1}(U)\) is \((1, 2)^* - \alpha^* -\) open in X, for every \(\sigma_{1,2}\)-open set \(U\) in Y.

**Proof.** Let \(f\) be an \((1, 2)^* - \alpha^* -\) continuous function and \(U\) be an \(\sigma_{1,2}\)-open set in Y. Then \(f^{-1}(U^c)\) is \((1, 2)^* - \alpha^* -\) closed in X. But \(f^{-1}(U^c) = [f^{-1}(U)]^c\) and hence \(f^{-1}(U)\) is \((1, 2)^* - \alpha^* -\) open in X. Conversely, \(f^{-1}(U)\) is \((1, 2)^* - \alpha^* -\) open in X for every \(\sigma_{1,2}\)-open set \(U\) in Y. \(U^c\) is \(\sigma_{1,2}\)-closed set \(U\) in Y. Then \(f^{-1}(U^c)\) is
(1, 2)* – α*–closed in X. But \([f^{-1}(U)]^c = f^{-1}(U^c)\) and hence \(f^{-1}(U^c)\) is (1, 2)* – α*–closed in X. Therefore, \(f\) is (1, 2)* – α*–continuous function.

**Theorem 3.28:** Let \(f: X \rightarrow Y\) and \(g: Y \rightarrow Z\) be two functions. Then
1. \(gof: X \rightarrow Z\) is (1, 2)*-α*-continuous, if \(g\) is (1, 2)*-continuous and \(f\) is (1, 2)*-α*-continuous function.
2. \(gof: X \rightarrow Z\) is (1, 2)*- α*-irresolute, if \(g\) is (1, 2)*- α*-irresolute and \(f\) is (1, 2)*- α*-continuous.
3. \(gof: X \rightarrow Z\) is (1, 2)*- α*- continuous, if \(g\) is (1, 2)*- α*-continuous and \(f\) is (1, 2)*- α*-irresolute.

**Proof.** The proof follows from the definitions.

**Lemma 3.29:** The product of two (1, 2)* - α*-open sets is (1, 2)* - α*-open sets in the product space.

**Proof.** Let \(A\) and \(B\) be (1, 2)* – α*-open sets of two space \(X\) and \(Y\) respectively and \(V = A \times B \subseteq X \times Y\). Let \(F \subseteq V\) be a \((1, 2)* – \alpha – closed in X \times Y\), then there exists two \((1, 2)* – \alpha – closed sets \(F_1 \subseteq A, F_2 \subseteq B\). So, \(F_1 \subseteq \tau_{1,2} – \text{int}(A), F_2 \subseteq \tau_{1,2} – \text{int}(B)\). Hence, \(F_1 \times F_2 \subseteq A \times B\) and \(F_1 \times F_2 \subseteq \tau_{1,2} – \text{int}(A) \times \tau_{1,2} – \text{int}(B)\). Therefore, \(A \times B\) is (1, 2)* – α*-open set of a space \(X \times Y\).

**Theorem 3.30:** Let \(f: X \rightarrow Y\) be a function. Then the following statements are equivalent.
1. \(f\) is (1, 2)* – α*-irresolute function.
2. For \(x \in X\) and any (1, 2)* – α*-closed set \(V\) of \(Y\) containing \(f(x)\), there exists an (1, 2)* – α*-closed set \(U\) such that \(x \in U\) and \(f(U) \subseteq V\).
3. Inverse image of every (1, 2)* – α*-open set of \(Y\) is (1, 2)* – α*-open in \(X\).

**Proof.** [1] \(\rightarrow\) [2]: Let \(V\) be an (1, 2)* – α*-closed set of \(Y\) and \(f(x) \in V\). Since \(f\) is (1, 2)* – α*-irresolute, \(f^{-1}(V)\) is (1, 2)* – α*-closed in \(X\) and \(x \in f^{-1}(V)\). Put \(U = f^{-1}(V)\). Then, \(x \in U\) and \(f(U) \subseteq V\).

[2] \(\rightarrow\) [1]: Let \(V\) be an (1, 2)* – α*-closed set of \(Y\) and \(x \in f^{-1}(V)\). Then \(f(x) \in V\). Therefore, by [2], there exists an (1, 2)* – α*-closed set \(U_x\) such that \(x \in U_x\) and \(f(U_x) \subseteq V\). Hence \(x \in U_x \subseteq f^{-1}(V)\). This implies that, \(f^{-1}(V)\) is a union of \((1, 2)* – \alpha* – closed sets of \(X\). Thus, \(f^{-1}(V)\) is (1, 2)* – α*-closed set. This shows that, \(f\) is (1, 2)* – α*-irresolute function.

[2] \(\leftrightarrow\) [3]: It is Obvious.

**References**


