International Journal of Pure and Applied Mathematics
Volume 117 No. 13 2017, 69-79
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
Special Issue

ijpam.eu

Schur Convexities and Concavities of Generalized Heron Means

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Abstract

In this paper, the various kinds of Schur convexities and concavities of generalized Heron mean, similar product type means and their dual forms in two variables are discussed using strong mathematical induction by grouping of terms.

AMS Subject Classification: 26D10, 26D15.

Key Words and Phrases: Heron mean, generalized Heron means, Schur concavity and convexity

1 Introduction

For positive numbers a, b, let

$$I = I(a,b) = \begin{cases} \exp\left[\dfrac{b\ln b - a\ln a}{b-a} - 1\right], & a < b; \\ a, & a = b; \end{cases} \tag{1}$$

$$L = L(a,b) = \begin{cases} \dfrac{a-b}{\ln a - \ln b}, & a \neq b; \\ a, & a = b; \end{cases} \tag{2}$$

and

$$H = H(a,b) = \frac{a + \sqrt{ab} + b}{3}. \tag{3}$$

These are respectively called the Identric, Logarithmic and Heron means. In [5, 25, 26], V. Lokesha et al. studied extensively and obtained some remarkable results on the weighted Heron mean, the weighted Heron dual mean and the weighted product type means and its monotonicities. Shi et al.[18], discussed the Schur-convexity and Schur-geometric-convexity of a further generalization of the Heronian means given by

$$H_{p,w}(a,b) = \begin{cases} \left(\frac{a^p + w(ab)^{\frac{p}{2}} + b^p}{w+2}\right)^{\frac{1}{p}} & \text{if } p \neq 0 \\ \sqrt{ab} & \text{if } p = 0. \end{cases} \tag{4}$$

Recently, Li et al.[3] discussed the Schur-convexity and Schur-harmonic-convexity of the generalized Heronian means with two positive numbers. In [22, 24], Zhang et al. gave the generalizations of Heron mean, similar product type means and their dual forms. For two variables, the above means are as follows:

$$I(a,b;k) = \prod_{i=1}^{k}\left(\frac{(k+1-i)a + ib}{k+1}\right)^{\frac{1}{k}}, \quad I^*(a,b;k) = \prod_{i=0}^{k}\left(\frac{(k-i)a + ib}{k}\right)^{\frac{1}{k+1}}, \tag{5}$$

and

$$H(a,b;k) = \frac{1}{k+1}\sum_{i=0}^{k} a^{\frac{k-i}{k}} b^{\frac{i}{k}}, \quad H^*(a,b;k) = \frac{1}{k}\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}} b^{\frac{i}{k+1}}, \tag{6}$$

where k is a natural number. Authors have proved that $H(a,b;k)$ and $I^*(a,b;k)$ are monotonic decreasing functions and $H^*(a,b;k)$ and $I(a,b;k)$ are monotonic increasing functions with k and also established the following limiting values of these means.

$$\lim_{k\to+\infty} I(a,b;k) = \lim_{k\to+\infty} I^*(a,b;k) = I(a,b),$$

and

$$\lim_{k\to+\infty} H(a,b;k) = \lim_{k\to+\infty} H^*(a,b;k) = L(a,b).$$

The Schur convex function was introduced by I Schur, in 1923 and it has many important applications in analytic inequalities. In 2003 X.M. Zhang proposed the concept of "Schur-Harmonically convex function" which is an extension of "Schur-Convexity function". Schur-geometrically convexity for different means are discussed in [16, 23]. The detailed discussion on convexity and Schur convexity can also be found in ([2]-[14]).

2 Definitions and Lemmas

In this section, we recall the definitions and lemmas which are essential to develop this paper.

Definition 2.1. *[6], [20] $\Omega \subseteq R^n$ is called symmetric set if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P.*
The function $\varphi : \Omega \to R$ is called symmetric if for every permutation matrix P, $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

Lemma 2.1. *[27] Let $\Omega \subseteq R^n$ be symmetric with non empty interior geometrically convex set Ω^0 and let $\varphi : \Omega \to R_+$ be continuous on Ω and differentiable in Ω^0. Then φ is Schur-geometrically convex(Schur-geometrically concave) on Ω if and only if φ is symmetric on Ω and*

$$(lnx_1 - lnx_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geq 0(\leq 0), \tag{7}$$

Then φ is Schur convex(Schur concave) on Ω if and only if φ is symmetric on Ω and

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0(\leq 0). \tag{8}$$

Then φ is Schur-harmonic convex(Schur-harmonic concave) on Ω if and only if φ is symmetric on Ω and

$$(x_1 - x_2)\left(x_1^2\frac{\partial\varphi}{\partial x_1} - x_2^2\frac{\partial\varphi}{\partial x_2}\right) \geq 0(\leq 0), \tag{9}$$

holds for any $x = (x_1, x_2, ..., x_n) \in \Omega^0$.

3 Main Result

In this section, the various kinds of Schur convexities and concavities of generalized Heron mean, similar product type means and their dual forms in two variables are discussed using strong mathematical induction[17] with grouping of terms.

Theorem 1. *Let a, b be positive real numbers and k be non-negative integer. Then generalized Heron mean $H(a, b; k)$ is*

1. *Schur-geometric convex(concave) for all values of k and $a \geq (\leq)b$.*

2. *Schur concave(convex) for all values of k and $a \leq (\geq)b$.*

3. *Schur-harmonic convex(concave) for all values of k and $a \geq (\leq)b$.*

Proof: The proof is established by discussing the following three cases.

Case(i). For $a > b > 0$ and k be non-negative integer, we have the generalized Heron mean,

$$H(a,b;k) = \frac{1}{k+1}\sum_{n=0}^{k} a^{\frac{k-n}{k}} b^{\frac{n}{k}}. \tag{10}$$

Let us differentiate partially w.r.t a and multiply by a, then we have

$$a\frac{\partial H}{\partial a} = \frac{1}{k+1}\sum_{n=0}^{k} \frac{(k-n)}{k} a^{\frac{k-n}{k}} b^{\frac{n}{k}} \tag{11}$$

Similarly,

$$b\frac{\partial H}{\partial b} = \frac{1}{k+1}\sum_{n=0}^{k} \frac{n}{k} a^{\frac{k-n}{k}} b^{\frac{n}{k}} \tag{12}$$

Consider,

$$(lna - lnb)\left(a\frac{\partial H}{\partial a} - b\frac{\partial H}{\partial b}\right) = \frac{(lna - lnb)}{k+1}\sum_{n=0}^{k} \frac{(k-2n)}{k} a^{\frac{k-n}{k}} b^{\frac{n}{k}} \tag{13}$$

$$(lna - lnb)\left(a\frac{\partial H}{\partial a} - b\frac{\partial H}{\partial b}\right) = [\Delta]\,[\Theta]\,. \tag{14}$$

Where,

$$\Delta = \frac{(lna - lnb)}{k+1} \quad \text{and} \quad \Theta = \sum_{n=0}^{k} \frac{(k-2n)}{k} a^{\frac{k-n}{k}} b^{\frac{n}{k}}$$

Clearly $\Delta \geq 0$.

Now, we shall prove that $\Theta > 0$, for all positive integral values of k, by strong mathematical induction.

For $k = 1$,

$$\Theta = a - b = 2sinht > 0$$

$$where \quad a = e^{t} \quad and \quad b = e^{-t}, \quad then \quad for \quad all \quad t \geq 0 \quad and \quad a > b$$

For $k = 2$,

$$\Theta = a - b = 2sinht > 0$$

For $k = 3$,

$$\Theta = \left[2sinht + \frac{2}{3}sinh(t/3)\right] > 0$$

For $k = 4$,

$$\Theta = [2sinht + sinh(t/2)] > 0$$

In general,

$$\Theta = a + \left(\frac{k-2}{k}\right) a^{\frac{k-1}{k}} b^{\frac{1}{k}} + \left(\frac{k-4}{k}\right) a^{\frac{k-2}{k}} b^{\frac{2}{k}} + ...$$

$$-\left[b + \left(\frac{k-2}{k}\right) a^{\frac{1}{k}} b^{\frac{k-1}{k}} + \left(\frac{k-4}{k}\right) a^{\frac{2}{k}} b^{\frac{k-2}{k}} ....\right]$$

From the above arguments we have two generalized subcases as follows:

Subcase(i). When k is even, Θ contains odd number of terms and middle term of Θ is zero, thus Θ contains only even number of terms on grouping first term and k^{th} term and second term and $(k-1)^{th}$ term and so on, shows that Θ is positive .

Subcase(ii). When k is odd, Θ contains only even number on grouping first term and k^{th} term and second term and $(k-1)^{th}$ term and so on, shows that Θ is positive .

Thus we conclude that $\Theta > 0$ for all integral values of k. Hence

$$(lna - lnb)\left(a\frac{\partial H}{\partial a} - b\frac{\partial H}{\partial b}\right) = [\Delta]\,[\Theta] > 0. \tag{15}$$

Case(ii). For $a > b > 0$ and k be non-negative integer, we have the generalized Heron mean,

$$H(a,b;k) = \frac{1}{k+1}\sum_{n=0}^{k} a^{\frac{k-n}{k}} b^{\frac{n}{k}} \tag{16}$$

Let us differentiate partially w.r.t a, then we have

$$\frac{\partial H}{\partial a} = \frac{1}{k+1}\sum_{n=0}^{k} \frac{(k-n)}{k} a^{\frac{k-n}{k}-1} b^{\frac{n}{k}} \tag{17}$$

Similarly,

$$\frac{\partial H}{\partial b} = \frac{1}{k+1}\sum_{n=0}^{k} \frac{n}{k} a^{\frac{k-n}{k}} b^{\frac{n}{k}-1} \tag{18}$$

Consider,

$$(a-b)\left(\frac{\partial H}{\partial a} - \frac{\partial H}{\partial b}\right) = \frac{(a-b)}{(k+1)ab}\sum_{n=0}^{k} \frac{((k-n)b - na)}{k} a^{\frac{k-n}{k}} b^{\frac{n}{k}} \tag{19}$$

$$(a-b)\left(\frac{\partial H}{\partial a} - \frac{\partial H}{\partial b}\right) = [\Delta]\,[\Theta]\,. \tag{20}$$

Where,

$$\Delta = \frac{(a-b)}{(k+1)ab} \quad \text{and} \quad \Theta = \sum_{n=0}^{k} \frac{((k-n)b - na)}{k} a^{\frac{k-n}{k}} b^{\frac{n}{k}}$$

Clearly $\Delta \geq 0$.

Now, we shall prove that $\Theta \leq 0$, for all positive integral values of k, by strong mathematical induction.

For $k = 1$,

$$\Theta = ab - ab = 0$$

$$where \quad a = e^t \quad and \quad b = e^{-t}, then \quad for \quad all \quad t \geq 0 \quad and \quad a > b$$

For $k = 2$,

$$\Theta = \frac{(b-a)}{2}(ab)^{\frac{1}{2}} = -2sinht < 0$$

For $k = 3$,

$$\Theta = -\left[\frac{8}{3}sinh(2t/3) \quad cosh^2(t/3)\right] < 0$$

For $k = 4$,

$$\Theta = \left[-sinht - \frac{1}{2}sinh(3t/2) - \frac{3}{2}sinh(t/2)\right] < 0$$

In general,

$$\Theta = \left[ab + \left(\frac{(k-1)b - a}{k}\right) a^{\frac{k-1}{k}} b^{\frac{1}{k}} + \left(\frac{(k-2)b - 2a}{k}\right) a^{\frac{k-2}{k}} b^{\frac{2}{k}} + ...\right]$$
$$- \left[ab + \left(\frac{(k-1)a - b}{k}\right) a^{\frac{1}{k}} b^{\frac{k-1}{k}} + ..\right]$$

From the above arguments we have two generalized subcases as follows:

Subcase(i). When k is even, Θ contains odd number of terms and one term of Θ is zero, thus Θ contains only even number of terms on grouping first term and k^{th} term and second term and $(k-1)^{th}$ term and so on, shows that Θ is negative.

Subcase(ii). When k is odd, Θ contains even number of terms and one term of Θ is zero, thus Θ contains only odd number of terms on grouping first term and k^{th} term and second term and $(k-1)^{th}$ term and so on, shows that Θ is negative.

Thus we conclude that $\Theta \leq 0$ for all integral values of k. Hence

$$(a-b)\left(\frac{\partial H}{\partial a} - \frac{\partial H}{\partial b}\right) = [\Delta]\,[\Theta] \leq 0. \tag{21}$$

Case(iii). For $a > b > 0$ and k be non-negative integer, we have the generalized Heron mean,

$$H(a, b; k) = \frac{1}{k+1}\sum_{n=0}^{k} a^{\frac{k-n}{k}} b^{\frac{n}{k}} \tag{22}$$

Let us differentiate partially w.r.t a and multiply by a^2, then we have

$$a^2\frac{\partial H}{\partial a} = \frac{1}{k+1}\sum_{n=0}^{k} \frac{(k-n)}{k} a^{\frac{k-n}{k}+1} b^{\frac{n}{k}} \tag{23}$$

Similarly,

$$b^2\frac{\partial H}{\partial b} = \frac{1}{k+1}\sum_{n=0}^{k} \frac{n}{k} a^{\frac{k-n}{k}} b^{\frac{n}{k}+1} \tag{24}$$

Consider,

$$(a-b)\left(a^2\frac{\partial H}{\partial a} - b^2\frac{\partial H}{\partial b}\right) = \frac{(a-b)}{k+1}\sum_{n=0}^{k}\left(\frac{(k-n)a - nb}{k}\right) a^{\frac{k-n}{k}} b^{\frac{n}{k}} \tag{25}$$

$$(a-b)\left(a^2\frac{\partial H}{\partial a} - b^2\frac{\partial H}{\partial b}\right) = [\Delta]\,[\Theta]\,. \tag{26}$$

Where,

$$\Delta = \frac{(a-b)}{k+1} \quad \text{and} \quad \Theta = \sum_{n=0}^{k} \left(\frac{(k-n)a - nb}{k} \right) a^{\frac{k-n}{k}} b^{\frac{n}{k}}$$

Clearly $\Delta \geq 0$.

Now, we shall prove that $\Theta > 0$, for all positive integral values of k, by strong mathematical induction.

For $k = 1$,

$$\Theta = a^2 - b^2 = 2sinh2t > 0$$

$$where \quad a = e^t \quad and \quad b = e^{-t}, then \quad for \quad all \quad t \geq 0 \quad and \quad a > b$$

For $k = 2$,

$$\Theta = a^2 - b^2 + \frac{(a-b)}{2}(ab)^{\frac{1}{2}} = 2sinh2t + sinht > 0$$

For $k = 3$,

$$\Theta = 2sinh2t + \frac{1}{3}\left[2sinh(2t/3) + 4sinh(4t/3)\right] > 0$$

For $k = 4$,

$$\Theta = 2sinh2t + sinht + \frac{1}{4}\left[2sinh(t/2) + 6sinh(3t/2)\right] > 0$$

In general,

$$\Theta = \left[a^2 + \left(\frac{(k-1)a-b}{k}\right) a^{\frac{k-1}{k}} b^{\frac{1}{k}} + \left(\frac{(k-2)a-2b}{k}\right) a^{\frac{k-2}{k}} b^{\frac{2}{k}} + ...\right]$$
$$- \left[b^2 + \left(\frac{(k-1)b-a}{k}\right) a^{\frac{1}{k}} b^{\frac{k-1}{k}} + ...\right]$$

From the above arguments we have two generalized subcases as follows:

Subcase(i). When k is even, then Θ contains only odd number of terms on grouping first term and k^{th} term and second term and $(k-1)^{th}$ term and so on, shows that Θ is positive.

Subcase(ii). When k is odd, Θ contains only even number of terms on grouping first term and k^{th} term and second term and $(k-1)^{th}$ term and so on, shows that Θ is positive .

Thus we conclude that $\Theta > 0$ for all integral values of k.

Hence

$$(a-b)\left(a^2\frac{\partial H}{\partial a} - b^2\frac{\partial H}{\partial b}\right) = [\Delta]\,[\Theta] > 0. \tag{27}$$

with similar arguments follows the proof of the following theorems.

Theorem 2. *Let a , b be positive real numbers and k be non- negative integer. Then generalized dual form of Heron mean $H^*(a, b; k)$ is*

1. *Schur-geometrically convex(concave) for all values of k and $a \geq (\leq)b$.*
2. *Schur concave(convex) for all values of k and $a \leq (\geq)b$.*

3. Schur-harmonic convex(concave) for all values of k *and* $a \geq (\leq) b$.

Theorem 3. *Let* a, b *be positive real numbers and* k *be non- negative integer. Then generalized Heron mean similar to product type* $I(a, b; k))$ *is*

1. *Schur-geometrically convex(concave) for all values of* k *and* $a \leq (\geq) b$.
2. *Schur convex(concave) for all values of* k *and* $a \leq (\geq) b$.
3. *Schur-harmonic convex(concave) for all values of* k *and* $a \geq (\leq) b$.

Theorem 4. *Let* a , b *be positive real numbers and* k *be non- negative integer. Then generalized dual form of Heron mean similar to product type* $I(a, b; k))$ *is*

1. *Schur-geometrically convex(concave) for all values of* k *and* $a \geq (\leq) b$.
2. *Schur concave(convex) for all values of* k *and* $a \leq (\geq) b$.
3. *Schur-harmonic convex(concave) for all values of* k *and* $a \geq (\leq) b$.

Acknowledgement

The authors are thankful to the referees for their valuable suggestions.

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