Proper Lucky Number of Torus Network

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Abstract

Let \( G(V,E) \) be a graph with vertex set \( V \) and edge set \( E \) and let \( f : V(G) \to N \) be a labeling defined in \( G \). Define the sum of neighbourhood of vertex \( v \) by \( s(v) = \sum_{u \in N(v)} f(u) \), where \( N(v) \) denotes the open neighbourhood of \( v \). A labeling \( f \) is a proper lucky labeling if \( f(u) \neq f(v) \) and \( s(u) \neq s(v) \) for all \( (u,v) \in E(G) \). The proper lucky number of \( G \), denoted by \( \eta_p(G) \) is the least positive integer \( k \) such that \( G \) has a proper lucky labeling with \( \{1, 2, \ldots, k\} \) as the set of labels. In this paper we determine proper lucky number of Torus network.

Keywords: Lucky labeling, Proper lucky number, Torus.

1 Introduction

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions [13]. Rosa, in the year 1967 introduced the concept of labeling and it was further developed by Graham and Sloane in 1980. The concept of labeling has much importance in graph theory as it is being used in various fields such as communication networks, coding theory, astronomy etc. Graph labeling is one of the most studied subjects in graph theory. It is an assignment of labels called colors to the vertices of a graph, subject to certain constraints. The problem of labeling offers numerous variants and established great significance at recent times. Graph labeling is used in various research areas of computer science such as networking, image segmentation, clustering, image capturing and data mining. The lucky labeling of graphs were studied by A. Ahai et al [2] and S. Akbari et al [3]. Let \( f : V(G) \to N \) be a labeling of the vertices of a graph by positive integers and \( s(v) \) denote the sum of labels of the neighbours of the vertex \( v \) in \( G \) then the
labeling is called lucky if the function $s$ is a proper coloring of $G$. The least positive integer $k$ for which a graph $G$ has a lucky labeling from the set $\{1, 2, \ldots, k\}$ is the lucky number of $G$, denoted by $\eta(G)$. Kins et al [10] obtained the lower bound of proper lucky number for any connected graph $G$ using clique number $\omega$.

**Definition 1.1.** [10] Let $G(V,E)$ be a graph with vertex set $V$ and edge set $E$. Let $f$ be a labeling defined in $G$. Define the sum of neighbourhood of vertex $v$ by $s(v) = \sum_{u \in N(v)} f(u)$, where $N(v)$ denotes the open neighbourhood of vertex of $v \in V$. A labeling $f$ is a proper lucky labeling if $f(u) \neq f(v)$ and $s(u) \neq s(v)$ for all $(u, v) \in E(G)$. The proper lucky number of $G$, denoted by $\eta_p(G)$ is the least positive integer $k$ such that $G$ has a proper lucky labeling with $\{1, 2, \ldots, k\}$ as the set of labels.

**Result 1.2.** [10] For any connected graph $G$, the chromatic number is less than or equal to proper lucky number i.e., $\chi \leq \eta_p$. For any connected graph $G$, let $\eta_p$ be its proper lucky number and $\omega$ be its clique number, then $\omega \leq \eta_p$.

2 Proper Lucky Number of Hexagonal Mesh

**Definition 2.1.** An $m \times n$ torus $TR(m,n)$ is defined as a graph with vertex set $V = \{(i,j); 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E = \{(i_1,j_1),(i_2,j_2); (i_2 = (i_1 + 1)(mod \ m) \land j_1 = j_2)V(i_1 = i_2 \land j_2((i_1 + 1)(mod \ n))\}$. The number of vertices in $TR(m,n)$ is $mn$ and the number of edges is $2mn$. It is 4-regular and its diameter is $\lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$.

Torus is bipartite if and only if all side lengths are even. They are Hamiltonian, regular and vertex symmetric. An one-dimensional torus is simply a circle or a ring. A two- dimensional torus network contains $mn$ nodes arranged in two dimension with $m,n$ nodes per dimension. We denote the $m \times n$ torus by $TR(m,n)$.

![Fig 1. The Torus TR₆×₆](image)

**Theorem 2.2.** The proper lucky number of even $TR_{n \times n}$ satisfies $\eta_p(TR_{n \times n}) = 2$.

**Proof.** Let $G$ be a torus $TR_{n \times n}$ of dimension $n$. Define a mapping $f : V(G) \rightarrow \{1, 2\}, \forall v_{ij} \in V$ as follows.
\[ f(v_{2i-1,2j-1}) = 1, \quad i = 1, 2, \ldots, n/2, \quad j = 1, 2, \ldots, n/2. \]
\[ f(v_{2i,2j}) = 1, \quad i = 1, 2, \ldots, n/2, \quad j = 1, 2, \ldots, n/2. \]
\[ f(v_{2i-1,2j}) = 2, \quad i = 1, 2, \ldots, n/2, \quad j = 1, 2, \ldots, n/2. \]
\[ f(v_{2i,2j-1}) = 2, \quad i = 1, 2, \ldots, n/2, \quad j = 1, 2, \ldots, n/2. \]

**Claim:** To prove that \( f(u) \neq f(v) \)

*Subcase (i):* If \( v_{ij} \in V \) is not a boundary vertex then it is adjacent to the vertices \( v_{ij-1}, v_{ij+1}, v_{(i-1)j}, v_{(i+1)j} \).

Let \( f(v_{ij}) = 1 \), then its adjacent vertices receives the map 2 under \( f \). i.e. \( f(v_{ij-1}) = 2, f(v_{ij+1}) = 2, f(v_{(i-1)j}) = 2 \). Clearly the adjacent vertices of \( v_{ij} \) which are adjacent to each other does not receive the same map under \( f \). Similarly if \( f(v_{ij}) = 2 \) then its adjacent vertices receives the map 1 under \( f \) as discussed above.

*Subcase (ii):* If \( v_{ij} \in V \) is a boundary vertex then it is adjacent to the vertices \( v_{i+1,j}, v_{i-1,j}, v_{i,j+1}, v_{i,j-1} \).

Let \( f(v_{ij}) = 1 \), then its adjacent vertices receives the map 2 under \( f \) i.e. \( f(v_{i+1,j}) = 2, f(v_{i-1,j}) = 2, f(v_{i,j+1}) = 2, f(v_{i,j-1}) = 2 \). Clearly the adjacent vertices of \( v_{ij} \) which are adjacent to each other does not receive the same map under \( f \). Similarly if \( f(v_{ij}) = 2 \), then its adjacent vertices receives the map 1 under \( f \) as discussed above.

Clearly \( f(u) \neq f(v) \), for all \( (u, v) \in E(G) \). Hence the given labeling is a proper labeling.

Next we claim that the given mapping is a lucky labeling. That is, to prove \( s(u) \neq s(v) \), for all \( (u, v) \in E(G) \).

We obtain \( s(v_{ij}) \), the inner sum of labels over all neighbours of vertex \( v_{ij} \). Consider any vertex of \( TR_{nm} \). Let \( v(3i, 3j) \) be the vertex with four adjacent vertices say \( v(3i, 3j - 1), v(3i - 1, 3j), v(3i, 3j + 1), v(3i + 1, 3j) \). Hence its sum of neighbourhood are \( s(v_{3i,3j}) = f(v_{3i,3j-1}) + f(v_{3i-1,3j}) + f(v_{3i,3j+1}) + f(v_{3i+1,3j}) = 1 + 1 + 1 + 1 = 4 \).

Here we are taking \( v(3i, 3j + 2) \) the adjacent vertices of \( v_{3i,3j} \).

\[ s(v_{3i-2,3j}) = f(v_{3i-1,3j+1}) + f(v_{3i,3j+2}) + f(v_{3i+1,3j+1}) + f(v_{3i,3j}) = 2 + 2 + 2 + 2 = 8. \]

From the above cases we see that \( s(u) \neq s(v) \) for all \( uv \in (G) \). Similarly, we can prove other cases. Therefore \( \eta_p \leq 2 \). Since the clique number of \( TR_{n \times n} \) is 2, and by the Result 1.2, \( \eta_p \geq 2 \). Therefore \( \eta_p(TR_{n \times n}) = 2 \).
Theorem 2.3. If $n$ is odd, then the proper lucky number of $TR_{n\times n}$ satisfies $\eta_p(TR_{n\times n}) \leq 6$.

Proof. Consider the torus $TR_{n\times n}$, where $n$ is odd. Then partition the torus into four mesh $M_{[n/2] \times [n/2]}$, as $V_1, V_2, V_3$ and $V_4$, then by collecting the vertices in $[n/2]$-th row and column as $V_5$ and $V_6$. Let

$V_1 = \{v_{ij}, i = 1, 2, \ldots, [n/2], j = 1, 2, \ldots, [n/2]\}$,
$V_2 = \{v_{ij}, i = 1, 2, \ldots, [n/2], j = [n/2] + 1, \ldots n\}$,
$V_3 = \{v_{ij}, i = [n/2] + 1, \ldots, n, j = 1, 2, \ldots [n/2]\}$,
$V_4 = \{v_{ij}, i = [n/2] + 1, \ldots n, j = [n/2] + 1, \ldots n\}$,
$V_5 = \{v_{i[n/2]}, i = 1, 2, \ldots, n\}$ and
$V_6 = \{v_{[n/2]j}, j = 1, 2, \ldots n\}$.

After dividing the torus into mesh, two cases arises. When mesh is odd the top first and the bottom last receive the same labeling, and the other two receive the same labeling. When the mesh is even the labeling of first part is the copy of all other meshes.

From the centre vertices of the torus ($[n/2], [n/2]$) divide the row into upper and lower parts of ($[n/2], [n/2]$) and similarly divided the column. The second half of the row is the copy of upper part of column and vise versa. The lower part of column is the copy of first half of row.

Define a mapping $f : TR_{n\times n} \rightarrow N$ as follows.

$f(v_{2i-1,2j-1}) = 1, i = 1, 2, \ldots, \left(\frac{n - [n/2]}{2}\right), j = 1, 2, \ldots, \left(\frac{n - [n/2]}{2}\right)$

$f(v_{2i,2j}) = 1 i = 1, 2, \ldots, \left(\frac{n - [n/2]}{2}\right) - 1, j = 1, 2, \ldots, \left(\frac{n - [n/2]}{2}\right) - 1$.

$f(v_{2i-1,2j}) = 2, i = 1, 2, \ldots, \left(\frac{n - [n/2]}{2}\right), j = 1, 2, \ldots, \left(\frac{n - [n/2]}{2}\right) - 1$.

$f(v_{2i,2j}) = 2, i = 1, 2, \ldots, \left(\frac{n - [n/2]}{2}\right) - 1, j = 1, 2, \ldots, \left(\frac{n - [n/2]}{2}\right)$.

$f(v_{2i-1,[n/2]}) = 3, i = 1, 2, \ldots, \left(\frac{n - [n/2]}{2}\right)$.

$f(v_{2i,[n/2]}) = 4, i = 1, 2, \ldots, \left(\frac{n - [n/2]}{2}\right) - 1$.

$f(v_{[n/2],2j-1}) = 4, i = 1, 2, \ldots, \left(\frac{n - [n/2]}{2}\right)$.
\[ f(v_{\lceil n/2 \rceil, 2j}) = 3, \ i = 1, 2, \ldots, \left( \frac{n - \lfloor n/2 \rfloor}{2} - 1 \right). \]

![Fig 4. Proper Lucky labeling of Torus TR_{7x7} and its sum of neighbourhood](image)

Rest of the proof is similar to theorem 2.2.

3 Conclusion

In this paper, we obtained the proper lucky number for torus network. Further, we investigate the problems in various interconnection networks such as butterfly, benes etc.

References


[10] Y.Kins, R.C. Thivyarathi, D.Antony Xavier, Proper Lucky number of Mesh derived architecture,


