Exponential Stability of Non-linear Stochastic Partial Integrodifferential Equations

A. Anguraj\textsuperscript{1} K. Ramkumar\textsuperscript{2}

\textsuperscript{1,2}Department of Mathematics
\textsuperscript{1,2}PSG College of Arts and Science, Coimbatore-14.
\textsuperscript{1}angurajpsg@yahoo.com, \textsuperscript{2}ramkumar_psg@yahoo.com

Abstract

We In this paper we proved the exponential stability in p-mean square with p>2 for the nonlinear stochastic partial integrodifferential equations by using the contraction mapping theorem.

Key Words: Exponential Stability, Mild Solution, Integrodifferential equations.

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1 Introduction

The study of stochastic partial integrodifferential equations in a separable Hilbert space has become an important area of investigation in the past two decades because of their applications to various problems arising in physics, biology, engineering, mechanics, economics and finance. The existence and stability of solutions of stochastic partial integrodifferential equations have been considered by many authors \cite{1, 2, 3, 4}. In \cite{23}, Caraballo and Liu established the exponential stability of the mild solution of a class stochastic partial differential equations with delays by using the Gronwall inequality. Liu and Shi \cite{15}, Liu \cite{13} have considered the exponential stability for stochastic partial function differential equations by means of the Rezuminkhin-type theorem. Moustapha dieye et.al. established the exponential stability of mild solutions for some stochastic partial integrodifferential equations with delays by using resolvent operators given in Grimmer \cite{5}.
In this present work, we study the exponential stability of mild solutions for the following nonlinear stochastic partial integrodifferential equations with delays

\[ dx(t) = \left[ Ax(t) + f(t, x(t)) + \int_0^t g(t - s)x(s)ds + B(x(\rho(t))) \right] dt + C(x(\tau(t)))dw(t) \]

\[ x(t) = \psi(t) \quad t \in [-h, 0] \] (1)

Where A generates a \( C_0 \)-semigroup on a separable Hilbert H, \( g(t) \) a closed linear operator on H with domain \( \mathcal{D}(A) \subset \mathcal{D}(G) \). Which is independent of t, \( t \geq 0 \), \( W(t) \) is a Wiener process on the separable Hilbert space U with covariance operator \( Q \in \mathcal{L}(U) \), \( f : R^+ \times H \to H \) is a Banach measurable function. \( B : H \to H, C : H \to \mathcal{L}(U, H), \rho, \tau : R^+ [-h, +\infty] (h \geq 0) \) are suitable delays functions, and \( \psi : [-h, 0] \times \Omega \to H \) is the initial datum.

As known, although the Lyapunov methods, is a powerful technique in proving the stability theorems, it is not so suitable in the delay case. A difficult is that mild solutions do not have stochastic differentials, so that one cannot apply the \( \text{ito} \) formula to them. Therefore, we encounter the same difficulties. It is well known that in the case without delays Lyapunov’s method is sufficient to obtain conditions for the stability of solutions. However, for delays differential equations even with constant delays this technique is no more available for instance see [10, 11, 20]. Caraballo developed in [4] a technique in order to obtain sufficient condition for the exponential stability of the strong solutions of a linear stochastic partial differential equation with delays. Using this technique Caraballo in [4] studied the stability methods of the equations (1) with \( f = 0, g = 0 \). Further Moustapha dieye [3] studied the existence and exponential stability of mild solutions for the equation studied in (1) with \( f = 0 \).

The paper is organized as follows. In section 2, we introduce preliminaries and some notations. In section 3, we study the exponential stability of the mild solution.

2 PRELIMINARIES

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) be a complete filtered probability space. Let \( \mathcal{L}(X) \) denotes the Bounded linear operator from a Banach space X into itself. Now for a Wiener process the probability space and having value in a separable Hilbert space u, one can construct \( w(t) \) as follows,

\[ w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n \beta_n(t)} e_n \quad t \geq 0, \]

where \( \beta_n(t) \) \((n = 1, 2, 3,...)\) is a sequence of real valued standard Brownian motions mutually independent on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \). Here \( \lambda_n \geq 0 \) \((n = 1, 2, 3,...)\) are positive real numbers such that \( \sum_{n=1}^{\infty} \lambda_n < \infty \), \((e_n)_{n \geq 1}\) is a complete orthonormal basis in U, and \( Q \in \mathcal{L}(U) \) is the incremental covariance operator of the W which is a symmetric nonnegative trace class operator defined by \( Qe_n = \lambda_n e_n \) \( n = 1, 2, 3,... \) Let \( L_0^2 = L_2(U_0, H) \) denotes the space of all Hilbert-schmidt operator from \( U_0 = Q^{1/2}(U) \)
to H which is a separable Hilbert space, endowed with the norm

\[ \| \varphi \|_2^2 = tr(\varphi Q \varphi^*) \]
Let $\varphi : (0, +\infty) \to L^0_2$ be a predictable $F_t$-adapted process such that
\[
\int_0^t E \|\varphi(s)\|_2^2 \, ds < \infty
\]
for $t > 0$.

Define the $H$-valued stochastic integral $\int \varphi(s) \, dw(s)$. Which will be a continuous square integrable martingale. For more details on stochastic integral, we refer to [24]. Let $Y$ the Banach space $D(A)$ equipped with the graph norm defined by $|y|_Y = |AY| + |Y|$ for $y \in Y$. the notations $C(\mathbb{R}^+; Y)$ we consider the following Cauchy problem we use the resolvent operator $R(t)$ given in [5] to define our solution.

$$\begin{cases}
V' = AV(t) + \int_0^t g(t-s)v(s) \, ds & \text{for } t \geq 0 \\
v(0) = v_0 \in H
\end{cases}$$

(2)

**Definition 1.** [5] A resolvent operator for equation (2) is a bounded linear, operator-valued function $R(t) \in \mathcal{L}(H)$ for $t \geq 0$, having the following properties.

1. $R(0) = I$ and $\|R(t)\|_{\mathcal{L}(H)} \leq Ne^{\eta t}$ for some constants $N > 0$ and $\eta \in (\mathbb{R})$.

2. For each $x \in H$, $R(t)x$ is strongly continuous for $t \geq 0$.

3. $R(t) \in \mathcal{L}(Y)$ for $t \geq 0$. For $x \in Y, R(.)x \in C^1(\mathbb{R}^+; H) \bigcap C(\mathbb{R}^+; Y)$ and

$$R'(t)x = AR(t)x + \int_0^t g(t-s)R(s)x \, ds$$

$$= R(t)Ax + \int_0^t R(t-s)g(s)x ds \quad \text{for } t \geq 0$$

Now we assume the following hypothesis:

$(H_1)$. For all $t \geq 0$, $g(t)$ is closed linear operator from $D(A)$ to $H$ and $g(t) \in \mathcal{L}(Y, H)$. For any $y \in Y$, the map $t \to g(t)y$ is bounded, differentiable and the derivative $t \to g'(t)y$ is bounded uniformly continuous on $\mathbb{R}^+$.

**Theorem 2.** [5] Assume that $(H_1)$ hold. Then there exists a unique resolvent operator of the Cauchy problem (2). In the following, we give some results for the existence of solutions for the following integrodifferential equation.

$$\begin{cases}
v' = Av(t) + \int_0^t g(t-s)v(s) \, ds + q(t) & \text{for } t \geq 0 \\
v(0) = v_0 \in H
\end{cases}$$

(3)

where $q : \mathbb{R}^+ \to H$ is a continuous function.

**Definition 3.** A continuous function $v : \mathbb{R}^+ \to H$ is said to be a strict solution of equation (3) if $v \in (\mathbb{R}^+; H) \bigcap C(\mathbb{R}^+; Y), v$ satisfies (3) for $t \geq 0$.

**Theorem 4.** [5] Assume that $H_1$ hold. if $v$ is a strict solution of equation (3) Then

$$v(t) = R(t)v_0 + \int_0^t R(t-s)g(s)ds \quad \text{for } t \geq 0$$
Definition 5. A mild solution of the integrodifferential equation (1) for \( T > 0 \) is a stochastic process \( x : [-h, T] \to H \) defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that \( x \) is \( \mathcal{F}_t \)-adapted predictable on \([-h, 0]\), satisfies with probability one \( \int_{-h}^T \|x(t)\|_H^2 \, dt < \infty \) and the following expression

\[
x(t) = R(t)\varphi(0) + \int_0^t R(t-s)f(s, x(s))\, ds + \int_0^t R(t-s)B(x(\rho(s)))\, ds + \int_0^t R(t-s)C(x(\tau(s)))\, dw(s)
\]

\( x(t) = \varphi(t) \) for \( t \in [-h, 0] \)

Let \( I = [-h, T] \) and denote \( \beta(I, H) \equiv \beta \) the space of \( \mathcal{F}_t \)-adapted predictable random process with values in the Hilbert space \( H \) satisfying \( \sup \mathbb{E}\|x(t)\|_H^2, t \in I < \infty \).

In order to set our problem, make the following hypothesis:

\( (H_5) \). There exists \( g > 0 \) and \( M \geq 1 \) such that \( (R(t))_{t \geq 0} \) the resolvent operator of equation (2) satisfies

\[
\|R(t)\|_{L(H)} \leq Me^{-\gamma t} \quad \text{for} \ t \geq 0
\]

\( (H_6) \). The functions \( f, A \) and \( C \) are Lipschitz continuous, that is there exists positive constants \( L_1, L_2 \) and \( L_3 \) such for every \( x, y \in H \) the following conditions are satisfied

\[
\|f(x) - f(y)\|_H \leq L_1 \|x - y\|_H \quad (5)
\]

\[
\|B(x) - B(y)\|_H \leq L_2 \|x - y\|_H \quad (6)
\]

\[
\|C(x) - C(y)\|_H \leq L_3 \|x - y\|_H \quad (7)
\]

\( (H_7) \). The delay functions \( \rho, \tau : [0, +\infty) \to [-h, +\infty), h > 0 \) are continuously differentiable. moreover, we assume that

\[
\rho'(t) \geq 1, \tau'(t) \geq 1, -h \leq \rho(t) \leq t \quad (8)
\]

and \( \tau(t) \leq t \quad \text{for} \ t \geq 0 \)

\( (H_8) \). The initial datum \( \varphi : [-h, +0] \times \Omega \to H \) is such that \( \varphi(t) = f_0 \)-measurable for all \( t \in [-h, 0] \) and

\[
\mathbb{E}\|\varphi(0)\|_H^2 + \int_{-h}^0 \mathbb{E}\|\varphi\|_H^2 \, ds < +\infty. \quad (9)
\]

Example 1. Taking \( \rho(t) = t - h_1 \) and \( \tau(t) = t - h_2 \) with \( h_1, h_2 > 0 \). By setting \( h = \max(h_1, h_2) \), it follows that the hypothesis on \( \rho \) and \( \tau \) in \( (H_7) \) are satisfied.

Remark 1. From \( (H_6) \) it is clear that there exists \( D > 0 \) such that

\[
\|B(x)\|_H^2 + \|C(x)\|_H^2 \leq D^2 (1 + \|x\|_H^2) \quad (10)
\]

From \( (H_7) \), one can see that there exists \( k \geq 0 \) such that

\[
\rho^{-1}(t), \tau^{-1}(t) \leq t + k \quad \text{for} \quad t \geq -h \quad (11)
\]

In the rest of this discussion, we set for all \( t \in [-h, 0] \), \( f_t = f_0 \).

Lemma 1. \cite{24} For any \( r \geq 1 \) and for an \( \mathcal{L}_2^0 \)-predictable process \( \Phi(.) \) we have the following inequality

\[
\sup_{u \in [0, t]} \mathbb{E}\left( \int_0^u \Phi(\sigma) \, d\sigma \right)^{2r} \leq (r(2r - 1))^r \left( \int_0^t \left( \mathbb{E}\|\Phi(\sigma)\|_{L_H^2}^{2r} \right)^{\frac{r}{2}} \, d\sigma \right)^r, \quad t \in [0, T]
\]
3 EXPONENTIAL STABILITY

Definition 6. Let $p \geq 2$ be an integer. The mild solution $x^\psi(t)$ equation (1) is said to be globally exponentially asymptotically stable in the $p$-th mean if there exists $a > 0$ and $L \geq 1$ such that, for any mild solution of equation (1), $x^\psi(t)$ corresponding to an initial value $\phi$ with $E \|\phi(0)\|^p_H$ + $\int_0^t E \|\phi(s)\|^p_H ds < \infty$, the following inequality holds:

$$E \left\| x^\psi(t) - x^\phi(t) \right\|^p_H \leq Le^{-at} \|\psi - \phi\|^p_1, \quad t \geq 0 \quad (12)$$

where $\|\psi - \phi\|^p_1 = \max \left\{ E \|\psi(0) - \phi(0)\|^p_H, \int_0^t E \|\psi(s) - \phi(s)\|^p_H ds \right\}$.

Theorem 7. Suppose that the hypothesis $(H_1) - (H_5)$ are satisfied. Let $p \geq 2$ be an integer and let $x(t) \equiv x^\psi(t)$ and $y(t) \equiv y^\phi(t)$ be solution of equation (1) with initial values $\psi$ and $\phi$ respectively. Then, the following inequality holds:

$$E \left\| x(t) - y(t) \right\|^p_H \leq \beta e^{-(\gamma - \alpha)t} \|\psi - \phi\|^p_1, \quad t \geq 0 \quad (13)$$

where

$$\alpha = 3^{p-1}M^p(\sigma_1 + \sigma_2 + \sigma_3), \quad \beta = 3^{p-1}M^p(1 + \sigma_1 + \sigma_2 + \sigma_3),$$

$$\sigma_1 = \gamma^{1-p}L_1^p e^{\gamma k}, \quad \sigma_2 = c_p L_2^p \left[ \frac{p - 2}{2(p - 1)\gamma} \right]^{\frac{p-2}{2}}, \quad \sigma_3 = c_p L_2^p \left[ \frac{p - 2}{2(p - 1)\gamma} \right]^{\frac{p-2}{2}}$$

and $c_p = \left( \frac{\gamma(p-1)}{2} \right)^{\frac{p}{2}}$

Proof. Since $x(t)$ and $y(t)$ are two solutions of equations (1), we have

$$x(t) = R(t)\psi(0) + \int_0^t R(t-s)(f(s, x(s)))ds + \int_0^t R(t-s)B(x(\rho(s)))ds + \int_0^t R(t-s)C(x(\tau(s)))dw(s)$$

$$y(t) = R(t)\phi(0) + \int_0^t R(t-s)(f(s, y(s)))ds + \int_0^t R(t-s)B(y(\rho(s)))ds + \int_0^t R(t-s)C(y(\tau(s)))dw(s)$$

thus, it follows that

$$E \left\| x(t) - y(t) \right\|^p_H \leq 4^{p-1}E \|R(t)\psi(0) - \phi(0)\|^p_H$$

$$+ 4^{p-1}E \left\| \int_0^t R(t-s)(f(s, x(s)) - f(s, y(s)))ds \right\|^p_H$$

$$+ 4^{p-1}E \left\| \int_0^t R(t-s)(B(x(\rho(s))) - B(y(\rho(s)))) \right\|^p_H$$

$$+ 4^{p-1}E \left\| \int_0^t R(t-s)(C(x(\tau(s))) - C(y(\tau(s))))dw(s) \right\|^p_H$$

$$= \epsilon_1(t) + \epsilon_2(t) + \epsilon_3(t) + \epsilon_4(t) \quad (14)$$

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As before, we use holder’s inequality. Now,

\[
\epsilon_1(t) \leq 4^{p-1} E \| R(t) (\psi(0) - \varphi(0)) \|_H^p \\
\leq 4^{p-1} \left( M e^{-\gamma t} \right)^p \| \psi(0) - \varphi(0) \|_H^p \\
\leq 4^{p-1} M p e^{-\gamma t} \| \psi - \varphi \|_1^p
\]

\[
\epsilon_2(t) \leq 4^{p-1} E \left\| \int_0^t R(t - s) (f(s, x(s)) - f(s, y(s))) \, ds \right\|_H^p \\
\leq 4^{p-1} E \left( \int_0^t M e^{-\gamma(t-s)} \| f(s, x(s)) - f(s, y(s)) \|_H \, ds \right)^p \\
\leq 4^{p-1} M p E \left\{ \left\{ \int_0^t e^{-\gamma(p-1)(t-s)} \, ds \right\}^{p-1} \left\{ \int_0^t e^{-\gamma(t-s)} \| f(s, x(s)) - f(s, y(s)) \|_H \, ds \right\}^p \right\}^{1/p} \\
\leq 4^{p-1} M p \left\{ e^{-\gamma(t-s)} \right\}^{p-1} E \int_0^t e^{-\gamma(t-s)} \| f(s, x(s)) - f(s, y(s)) \|_H^p \, ds
\]

\[
\leq 4^{p-1} M p \left( \frac{1}{\gamma} \right)^{p-1} L_1^p e^{-\gamma(t-s)} \| x(\rho(s)) - y(\rho(s)) \|_H^p \, ds \\
\leq 4^{p-1} M p \left( \frac{1}{\gamma} \right)^{p-1} L_1^p e^{-\gamma(t-s)} \| x(\rho(s)) - y(\rho(s)) \|_H^p \frac{du}{\rho(\rho^{-1}(u))} \\
\leq 4^{p-1} M p \sigma_1 e^{-\gamma t} \| \psi - \varphi \|_1^p + 4^{p-1} M p \sigma_1 \int_0^t e^{-\gamma(t-u)} E \| x(u) - y(u) \|_H^p \, du
\]

where \( \sigma_1 = \gamma^{1-p} L_1^p e^{\gamma} \)

\[
\epsilon_3(t) \leq 4^{p-1} E \left\| \int_0^t R(t - s) [B(x(\rho(s))) - B(y(\rho(s)))] \, ds \right\|_H^p \\
\leq 4^{p-1} E \left( \int_0^t \| R(t - s) [B(x(\rho(s))) - B(y(\rho(s)))] \|_H \, ds \right)^p \\
\leq 4^{p-1} E \left( \int_0^t M e^{-\gamma(t-s)} \| B(x(\rho(s))) - B(y(\rho(s))) \|_H \, ds \right)^p \\
\leq 4^{p-1} M p E \left( \int_0^t e^{-\gamma(p-1)(t-s)} \, ds \right)^p \left\{ \int_0^t e^{-\gamma(t-s)} \| B(x(\rho(s))) - B(y(\rho(s))) \|_H \, ds \right\}^p \\
\leq 4^{p-1} M p \left( \frac{1}{\gamma} \right)^{p-1} L_2^p e^{-\gamma(t-s)} \| x(\rho(0)) - y(\rho(0)) \|_H^p \frac{du}{\rho(\rho^{-1}(u))} \\
\leq 4^{p-1} M p \sigma_2 e^{-\gamma t} \| \psi - \varphi \|_1^p \, du \\
+ 4^{p-1} M p \sigma_2 \int_0^t e^{-\gamma(t-u)} E \| x(u) - y(u) \|_H^p \, du
\]

\[
\epsilon(t) = \epsilon_1(t) + \epsilon_2(t) + \epsilon_3(t)
\]
where $\gamma^{1-p}L_p^p e^{\gamma k}$. Now from Lemma(1) and use Holder’s inequality,
\[
\epsilon_4(t) \leq 4^{p-1} E \left\| \int_0^t R(t-s) \left[ C \left( x(\tau(s)) - C(y(\tau(s))) \right) \right] dw(s) \right\|_H^{p}
\]
\[
\leq 4^{p-1} \sup_{u \in [0,t]} E \left\| \int_0^u R(t-s) \left[ C \left( x(\tau(s)) - C(y(\tau(s))) \right) \right] dw(s) \right\|_H^{2(p/2)}
\]
\[
\leq 4^{p-1} c_p \left( \int_0^t \left( E \left\| R(t-s) \left[ C \left( x(\tau(s)) - C(y(\tau(s))) \right) \right] \right\|_H^{p/2} ds \right)^{\frac{p}{2}} \right)^{\frac{p}{2}}
\]
\[
\leq 4^{p-1} c_p M^p L_3^p \left( \int_0^t \left( e^{-\gamma(t-s)} E \left\| x(\tau(s)) - y(\tau(s)) \right\|_H^{p} \right)^{\frac{p}{2}} ds \right)^{\frac{p}{2}}
\]
\[
\leq 4^{p-1} c_p M^p L_3^p \left[ \frac{p-2}{2(p-1)\gamma} \right]^{\frac{p-2}{p}} \int_0^t e^{-\gamma(t-s)} E \left\| x(\tau(s)) - y(\tau(s)) \right\|_H^{p} ds
\]
\[
\leq 4^{p-1} c_p M^p L_3^p \left[ \frac{p-2}{2(p-1)\gamma} \right]^{\frac{p-2}{p}} \int_0^t e^{-\gamma(t-s)} E \left\| x(\tau(s)) - y(\tau(s)) \right\|_H^{p} ds
\]
\[
\leq 4^{p-1} M^p \sigma_3 e^{-\gamma t} \left\| \psi - \phi \right\|_1^p du + 4^{p-1} M^p \sigma_3 \int_0^t e^{-\gamma(t-u)} E \left\| x(u) - y(u) \right\|_H^p du
\]
(19)

We remark if $p = 2$, the inequality (18) also hold with convention $0^0 = 1$. If follows by proceeding as we did previously,
\[
\epsilon_4(t) \leq 4^{p-1} c_p M^p L_3^p \left[ \frac{p-2}{2(p-1)\gamma} \right]^{\frac{p-2}{p}} \int_0^t e^{-\gamma(t-s)} E \left\| x(\tau(s)) - y(\tau(s)) \right\|_H^{p} ds
\]
\[
\leq 4^{p-1} M^p \sigma_3 e^{-\gamma t} \left\| \psi - \phi \right\|_1^p du + 4^{p-1} M^p \sigma_3 \int_0^t e^{-\gamma(t-u)} E \left\| x(u) - y(u) \right\|_H^p du
\]
(19)

where $\sigma_3 = c_p L_3^p \left[ \frac{p-2}{2(p-1)\gamma} \right]^{\frac{p-2}{p}}$ and $c_p = \left( \frac{p(p-1)}{2} \right)^{\frac{p}{2}}$. From the inequalities (15),(16),(17) and (19), one can see that the inequality (14) becomes
\[
E \left\| x(t) - y(t) \right\| \leq 4^{p-1} M^p (1 + \sigma_1 + \sigma_2 + \sigma_3) e^{-\gamma t} \left\| \psi - \phi \right\|_1^p du
\]
\[
+ 4^{p-1} M^p (\sigma_1 + \sigma_2 + \sigma_3) \int_0^t e^{-\gamma(t-u)} E \left\| x(u) - y(u) \right\|_H^p du, \quad t \geq 0
\]
(20)

That is,
\[
e^{\gamma t} E \left\| x(t) - y(t) \right\|_H^p \leq 4^{p-1} M^p (1 + \sigma_1 + \sigma_2 + \sigma_3) \left\| \psi - \phi \right\|_1^p du
\]
\[
+ 4^{p-1} M^p (\sigma_1 + \sigma_2 + \sigma_3) \int_0^t e^{\gamma t} E \left\| x(u) - y(u) \right\|_H^p du, \quad t \geq 0.
\]
(21)

Hence Gronwall’s inequality yields
\[
e^{\gamma t} E \left\| x(t) - y(t) \right\|_H^p \leq 4^{p-1} M^p (1 + \sigma_1 + \sigma_2 + \sigma_3) \left\| \psi - \phi \right\|_1^p \exp \left\{ 4^{p-1} M^p (\sigma_1 + \sigma_2 + \sigma_3) t \right\}, \quad t \geq 0
\]
(22)
That is,
\[ E\|x(t) - y(t)\|_H^p \leq \beta \|\psi - \phi\|_1^p e^{-(\gamma - \alpha)t}, \quad t \geq 0 \] (23)
Which complete the proof. \qed

**Corollary 8.** Under the hypothesis of the theorem (3) with \( \gamma > \alpha \), all the mild solutions of Equation (1) are globally exponentially asymptotically stable in the p-mean.

**References**


