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The tanh - coth Method for Soliton and Exact Solutions of the Sawada - Kotera Equation

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Abstract

In this paper, we consider Sawada - Kotera equation to obtain explicit exact soliton, compacton, periodic and travelling wave solutions by using $\tanh - \coth$ method with the help of Mathematica. This significant method for handling many integrable nonlinear evolution equations to produce various types of solitary wave solutions.

AMS Subject Classification: 35C08, 35Q51.

Key Words: Sawada - Kotera equation, $\tanh - \coth$ method, soliton, exact travelling wave solutions.

1 Introduction

In past decades many nonlinear evolution equations are extensively studied by various types of powerful methods[1, 2, 3]. Nonlinearity play the central role for mathematics, physics, engineering and scienctific fields to produce various types of solutions of nonlinear partial differential equations(NLPDE) for different methods. The following methods, that is the inverse scattering method[1], Miura transformation[3], Auto - Bäcklund transformation[4], similarity transformation[4, 5], Painlevé analysis[6], (G'/G) Expansion method[7], homogeneous balance method[8], Hirota bilinear formalism[2, 3, 9, 10, 11], homotopy analysis method[19], variational iteration method[20], etc. are commonly used to find solutions of NLPDE.

We consider the Sawada - Kotera[11] equation,

$$u_t + 45u^2u_x - 15u_xu_{xx} - 15uu_{xxx} + u_{xxxxx} = 0 \tag{1}$$

Equation (1) represents a nonlinear physical model equation. This significant model equation was studied in reference[9, 10, 11, 12, 13, 14, 15] for different approaches.

In section (3), we apply the $\tanh-\coth$[8, 9, 10, 13, 14, 15, 16, 17] method for Sawada - Kotera equation (1), formally obtain soliton, compacton, periodic and travelling wave solutions and given some of their illustrations. The new explicit soliton solution adequate previously known[20] numerical soliton solution and also produced new explicit solitary wave solutions.

2 The *tanh-coth* Method

The tanh - coth method discovered by Wazwaz [16].

A wave variable $\xi = x - ct$ converts any Partial Differential Equation (PDE)

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, ...) = 0, \tag{2}$$

to an Ordinary Differential Equation (ODE)

$$Q(u, u', u'', u''', ...) = 0. \tag{3}$$

Eq. (3) is then integrated as long as all terms contains derivatives where integration constants are considered zeros. The standard tanh method is developed by Malfliet[18], where the tanh is used as a new variable, since all derivatives of a tanh are represented by tanh itself. Introducing a new independent variable,

$$Y = \tanh(\mu\xi), \qquad \xi = x - ct, \tag{4}$$

where, μ is the wave number, leads to the change of derivatives:

$$\begin{aligned} \frac{d}{d\xi} &= \mu(1-Y^2)\frac{d}{dY}, \\ \frac{d^2}{d\xi^2} &= -2\mu^2 Y(1-Y^2)\frac{d}{dY} + \mu^2(1-Y^2)^2\frac{d^2}{dY^2}. \end{aligned} \tag{5}$$

The $\tanh-\coth$ method admits the use of the finite expansion

$$u(\mu\xi) = s(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \tag{6}$$

and

$$Y' = \mu(1 - Y^2). \tag{7}$$

where, M is a positive integer, in most cases, that will be determined by Homogeneous Balance Method (HBM).
Substituting (6) and (7) into the reduced ODE results in an algebraic equation in powers of Y.
Balancing the highest order the linear term with the highest order nonlinear term to determine the parameter M. We then collect all coefficients of powers of Y in the

resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters a_k, b_k, μ, and c. Having determined these parameters, we obtain an analytic solution $u(x,t)$ in a closed form. The solutions we obtain may be solitons, kink, compacton, peakon, cuspon, travelling wave and periodic solutions as well.

3 Sawada Kotera Equation

The Sawada Kotera Equation is

$$u_t + 45u^2u_x - 15u_xu_{xx} - 15uu_{xxx} + u_{xxxxx} = 0 \tag{8}$$

$$u(x,t) = u(\xi), \qquad \xi = x - ct \tag{9}$$

using (9) into (8) and integrating we get an ODE

$$-cu + 45\frac{u^3}{3} - 15uu'' + u'''' = 0 \tag{10}$$

From Eq. (10) balancing the nonlinear term u^3 with the higher order derivative u'''' we get the value $M = 2$ and hence,

$$u(x,t) = s(Y) = a_0 + a_1Y + a_2Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \tag{11}$$

Substituting (11) into (10), collecting the coefficients of each power of Y^i, setting each coefficient to zero.

$$\left.\begin{array}{r}60a_1b_1\mu^2+240a_2b_2\mu^2+45a_2b_1^2+90a_0a_1b_1+45a_1^2b_2+90a_0a_2b_2-a_0c\\-16a_2\mu^4-30a_0b_2\mu^2-30a_0a_2\mu^2+15a_0^3-16b_2\mu^4\end{array}\right\}=0 \tag{12}$$

$$\left.\begin{array}{r}150a_2b_1\mu^2-60a_1b_2\mu^2+45a_1^2b_1+90a_0a_2b_1+90a_1a_2b_2-a_1c+16a_1\mu^4\\+30a_0a_1\mu^2-30a_1a_2\mu^2+45a_0^2a_1\end{array}\right\}=0 \tag{13}$$

$$\left.\begin{array}{r}-30a_1b_1\mu^2-120a_2b_2\mu^2+90a_1a_2b_1+45a_2^2b_2-a_2c+30a_1^2\mu^2-30a_2^2\mu^2\\+120a_0a_2\mu^2+136a_2\mu^4+45a_0a_1^2+45a_0^2a_2\end{array}\right\}=0 \tag{14}$$

$$-90a_2b_1\mu^2+45a_2^2b_1-40a_1\mu^4-30a_0a_1\mu^2+150a_1a_2\mu^2+15a_1^3+90a_0a_1a_2=0 \tag{15}$$

$$-240a_2\mu^4-30a_1^2\mu^2+120a_2^2\mu^2-90a_0a_2\mu^2+45a_0a_2^2+45a_1^2a_2=0 \tag{16}$$

$$24a_1\mu^4-120a_1a_2\mu^2+45a_1a_2^2=0 \tag{17}$$

$$120a_2\mu^4-90a_2^2\mu^2+15a_2^3=0 \tag{18}$$

$$\left.\begin{array}{r}30a_0b_1\mu^2-60a_2b_1\mu^2+150a_1b_2\mu^2+45a_1b_1^2+45a_0^2b_1-b_1c+90a_0a_1b_2\\+90a_2b_1b_2+16b_1\mu^4-30b_1b_2\mu^2\end{array}\right\}=0 \tag{19}$$

$$\left.\begin{array}{r}-30a_1b_1\mu^2+120a_0b_2\mu^2-120a_2b_2\mu^2+45a_0b_1^2+45a_2b_2^2+45a_0^2b_2-b_2c\\+90a_1b_1b_2-30b_2^2\mu^2+136b_2\mu^4+30b_1^2\mu^2\end{array}\right\}=0 \tag{20}$$

$$-30a_0b_1\mu^2-90a_1b_2\mu^2+45a_1b_2^2+90a_0b_1b_2-40b_1\mu^4+150b_1b_2\mu^2+15b_1^3=0 \tag{21}$$

$$-90a_0b_2\mu^2+45a_0b_2^2-240b_2\mu^4-30b_1^2\mu^2+120b_2^2\mu^2+45b_1^2b_2=0 \tag{22}$$

$$24b_1\mu^4-120b_1b_2\mu^2+45b_1b_2^2=0 \tag{23}$$

$$120b_2\mu^4-90b_2^2\mu^2+15b_2^3=0 \tag{24}$$

Solving the resulting algebraic systems (12 - 24) with computer algebra software such as Mathematica, we obtain the following sets:

$$a_0 = -\frac{\sqrt{c}}{2}, \quad a_1 = 0, \quad a_2 = \frac{\sqrt{c}}{2}, \quad b_1 = 0, \quad b_2 = 0, \quad \mu = \frac{c^{\frac{1}{4}}}{2} \tag{25}$$

$$a_0 = -\frac{\sqrt{c}}{2}, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = \frac{\sqrt{c}}{2}, \quad \mu = \frac{c^{\frac{1}{4}}}{2} \tag{26}$$

$$a_0 = -\frac{\sqrt{c}}{4}, \quad a_1 = 0, \quad a_2 = \frac{\sqrt{c}}{8}, \quad b_1 = 0, \quad b_2 = \frac{\sqrt{c}}{8}, \quad \mu = \frac{c^{\frac{1}{4}}}{4} \tag{27}$$

$$a_0 = \frac{\sqrt{c}}{2}, \quad a_1 = 0, \quad a_2 = -\frac{\sqrt{c}}{2}, \quad b_1 = 0, \quad b_2 = 0, \quad \mu = \frac{\sqrt{-1}}{2}c^{\frac{1}{4}} \tag{28}$$

$$a_0 = \frac{\sqrt{c}}{2}, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = -\frac{\sqrt{c}}{2}, \quad \mu = \frac{\sqrt{-1}}{2}c^{\frac{1}{4}} \tag{29}$$

$$a_0 = \frac{\sqrt{c}}{4}, \quad a_1 = 0, \quad a_2 = -\frac{\sqrt{c}}{8}, \quad b_1 = 0, \quad b_2 = -\frac{\sqrt{c}}{8}, \quad \mu = \frac{\sqrt{-1}}{4}c^{\frac{1}{4}} \tag{30}$$

Substituting Eqs.(4) and (25-30) into Eq. (11), we obtain the following soliton, compacton, periodic and travelling wave solutions

$$u_1(x,t) = -\frac{\sqrt{c}}{2}\,\mathrm{sech}^2\left[\frac{1}{2}c^{\frac{1}{4}}(x-ct)\right], \quad c > 0 \tag{31}$$

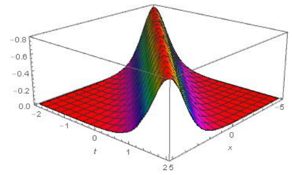
$$u_2(x,t) = \frac{\sqrt{c}}{2}\,\mathrm{csch}^2\left[\frac{1}{2}c^{\frac{1}{4}}(x-ct)\right], \quad c > 0 \tag{32}$$

$$u_3(x,t) = -\frac{\sqrt{c}}{8}\left(2 - \tanh^2\left[\frac{1}{4}c^{\frac{1}{4}}(x-ct)\right] - \coth^2\left[\frac{1}{4}c^{\frac{1}{4}}(x-ct)\right]\right), \quad c > 0 \tag{33}$$

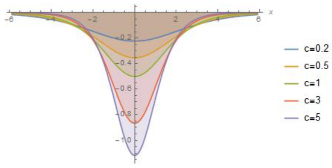
$$u_4(x,t) = \frac{\sqrt{c}}{2}\sec^2\left[\frac{1}{2}c^{\frac{1}{4}}(x-ct)\right], \quad c > 0 \tag{34}$$

$$u_5(x,t) = \frac{\sqrt{c}}{2}\csc^2\left[\frac{1}{2}c^{\frac{1}{4}}(x-ct)\right], \quad c > 0 \tag{35}$$

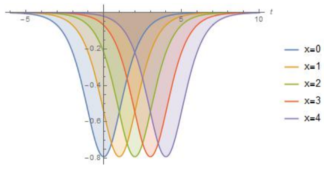
$$u_6(x,t) = \frac{\sqrt{c}}{8}\left(2 + \tan^2\left[\frac{1}{4}c^{\frac{1}{4}}(x-ct)\right] + \cot^2\left[\frac{1}{4}c^{\frac{1}{4}}(x-ct)\right]\right), \quad c > 0 \tag{36}$$



(a) $c = 2.5$, $-2 \leq t \leq 2$, $-5 \leq x \leq 5$



(b) different values of c with $t = 0$ and $-5.9 \leq x \leq 5.9$



(c) different values of x with $c = 2.5$ and $-6 \leq t \leq 10$

Figure 1: Soliton solution of $u_1(x,t)$

4 Conclusion

In this paper, we studied Sawada - Kotera equation for briefly. Many as possible solutions such as soliton, compacton, periodic and travelling wave solutions are obtained analytically by tanh - coth method with the help of Mathematica.

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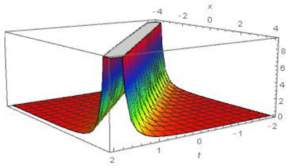
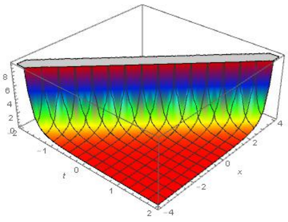
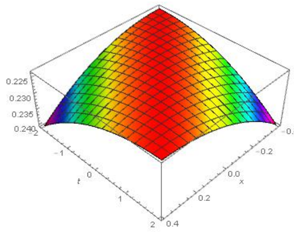


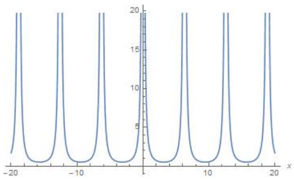
(a) $c = 2$, $-2 \le t \le 2$, $-4 \le x \le 4$



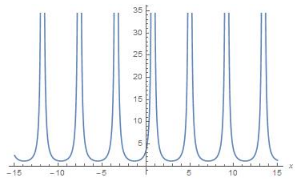
(b) $c = 2$, $-2 \le t \le 2$, $-4 \le x \le 4$



(c) $c = 0.2$, $-2 \le t \le 2$, $-0.4 \le x \le 0.4$



(d) $c = 1$, $t = 0$, $-20 \le x \le 20$



(e) $c = 5$, $t = 0$, $-15 \le x \le 15$

Figure 2: (a), (b) are Travelling wave solutions of $u_2(x,t)$, $u_3(x,t)$, (c) is Compacton solution of $u_4(x,t)$ and (d), (e) are Periodic solutions of $u_5(x,t)$, $u_6(x,t)$.

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