Narayana Prime Cordial Labeling of Graphs

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Abstract

In this paper we introduce a new graph labeling called Narayana prime cordial labeling of a graph \( G = (V, E) \) and we prove the existence of this labeling to the graphs viz., path, cycle and helm graph.

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Key Words and Phrases: Graphs, Narayana numbers, prime numbers, cordial labeling, Narayana prime cordial labeling.

1 Introduction

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions [1]. The basic notion of graph labeling is found in [7]. The vital application of labeled graphs can be found in science, engineering and technology and we refer [4] for the same. For graph labeling literature we refer [2]. We refer the text book Harary [3], for notations, concepts and terminology in graph theory.

The study of Narayana numbers [5] forms part of the branch of mathematics called number theory. In combinatorial mathematics, the Narayana numbers occur in various counting problems. The applications of Narayana numbers play a very important role in various fields of mathematics. Since Narayana numbers have many
applications, and graph labeling have often been motivated by practical problems, we are interested in Narayana labeling of graphs. Using Narayana number concept we introduce a new graph labeling known as Narayana prime cordial labeling of graphs. In this paper we prove that the graphs viz., path, cycle and helm graph admit a Narayana prime cordial labeling.

2 Preliminaries

In this section, we recall the definition of Narayana numbers, properties and application of Narayana numbers [6].

**Definition 1.** Let \( \mathbb{N}_0 \) be the set of non negative integers and let \( k, n \in \mathbb{N}_0 \). The Narayana numbers can be defined as

\[
N(n, k) = \frac{1}{n} \binom{n}{k} \left( \frac{n}{k+1} \right); 0 \leq k < n
\]

where \( \binom{n}{k} = \frac{n!}{(n-k)!k!} \).

The Narayana numbers were first studied by Macmahon and later rediscovered by Narayana. The Narayana numbers are closely related to the Catalan numbers [8]. That is,

\[
C_n = \frac{1}{n+1} \binom{2n}{n} \quad \text{and} \quad \sum_{k=0}^{n-1} N(n, k) = C_n
\]

where \( C_n \) is a Catalan number.

The Narayana numbers \( N(n, k) \) are given in the following triangular array in \( n \) rows and \( k \) columns such that the row sums are the Catalan numbers.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( k )</th>
<th>( 0 )</th>
<th>( 1 )</th>
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<th>( 3 )</th>
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<td>7</td>
<td>1</td>
<td>21</td>
<td>105</td>
<td>175</td>
<td>105</td>
<td>21</td>
<td>28</td>
<td>1</td>
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<td>8</td>
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<td>28</td>
<td>190</td>
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<td>490</td>
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<td>336</td>
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</tbody>
</table>

**Properties**

(i) If \( p \) is a prime number and if \( n = p^m \) for some \( m \in \mathbb{N}_0 \), then \( p \mid N(n, k) \) for \( 1 \leq k \leq n-2 \).

(ii) If \( p \) is a prime number and if \( n = p^m - 1 \) for some \( m \in \mathbb{N}_0 \), then \( p \nmid N(n, k) \) for \( 0 \leq k \leq n-1 \).
Application of Narayana numbers

We list out few applications of Narayana numbers:

(i) The Narayana numbers have been used in recent research on MIMO (Multiple input, multiple output) communication systems.

(ii) The Narayana numbers used in recent results on RNA Secondary Structure configurations and

(iii) They used in the partition of a graph in terms of trees.

3 New Results

In this section, we introduce a new graph labeling called Narayana prime cordial labeling of a graph \(G = (V, E)\) and we prove that the existence of this labeling to the graphs such as path, cycle and helm graph.

**Definition 2.** Let \(G(V, E)\) be a graph. An injective function \(f : V \rightarrow \mathbb{N}_0\) is said to be a Narayana prime cordial labeling of the Graph \(G\) if the induced edge function \(f^* : E \rightarrow \{0, 1\}\) satisfies the following conditions:

(i) For every \(uv \in E\)

\[f^*(uv) = 1 \text{ if } p|N(f(u), f(v)), \text{ where } f(u) > f(v) \text{ and } f(u) = p^m\]

for some \(m \in \mathbb{N}_0; 1 \leq f(v) \leq f(u) - 2\) where \(p\) is a prime number

\[= 1 \text{ if } p|N(f(v), f(u)), \text{ where } f(v) > f(u) \text{ and } f(v) = p^m\]

for some \(m \in \mathbb{N}_0; 1 \leq f(u) \leq f(v) - 2\) where \(p\) is prime number.

\[= 0 \text{ if } p \nmid N(f(u), f(v)), \text{ where } f(u) > f(v) \text{ and } f(u) = p^m - 1\]

for some \(m \in \mathbb{N}_0; 0 \leq f(v) \leq f(u) - 1\) where \(p\) is a prime number.

\[= 0 \text{ if } p \nmid N(f(v), f(u)), \text{ where } f(v) > f(u) \text{ and } f(v) = p^m - 1\]

for some \(m \in \mathbb{N}_0; 0 \leq f(u) \leq f(v) - 1\) where \(p\) is prime number.

(ii) \(|e_f(0) - e_f(1)| \leq 1\) where \(e_f(0)\) and \(e_f(1)\) denote respectively the number of edges with the label 0 and the number of edges with the label 1.

**Definition 3.** A graph \(G = (V, E)\) which admits a Narayana prime cordial labeling is called a Narayana prime cordial graph.

**Remark 4.** We call the Narayana prime cordial labeling of a graph as \(N\)-prime cordial labeling of a graph for simplicity in this paper.

**Theorem 5.** The path \(P_n\) admits a \(N\)-prime cordial labeling.

**Proof.** Let \(P_n\) be the path with \(n\) vertices and \(n - 1\) edges.

Let \(V = \{v_i|1 \leq i \leq n\}\) be the vertex set and \(E = \{v_iv_{i+1}|1 \leq i \leq n - 1\}\) be the edge set of the graph \(P_n\).

Define an injective function \(f : V \rightarrow \mathbb{N}_0\) such that

\[f(v_i) = 2^{i+1}, i \equiv 1 \text{ (mod 2)}\] and \(1 \leq i \leq n\)
\[ f(v_i) = 2^{i+1} - 1, \quad i \equiv 0 \pmod{2} \text{ and } 1 \leq i \leq n \]
and an induced edge function \( f^* : E \to \{0, 1\} \) as in the Definition 2.

**Case (i):** In this type of labeling pattern, when \( n \equiv 1 \pmod{2} \), there are an even number of edges in the graph \( P_n \) and the vertices of \( P_n \) receive the numbers \( p^m \) and \( p^r - 1 \), \( m, r \in \mathbb{N}_0 \) alternatively. By the properties of Narayana numbers \( n \frac{n}{2} \) number of edges receive the label 0 and \( n \frac{n}{2} \) number of edges receive the label 1.

That is, \( e_{f^*}(0) = \frac{n+1}{2} \) and \( e_{f^*}(1) = \frac{n-1}{2} \).
Therefore, the condition \( |e_{f^*}(0)| - e_{f^*}(1)| \leq 1 \) is satisfied.

**Case (ii):** when \( n \equiv 0 \pmod{2} \) there are an odd number of edges in \( P_n \). Therefore obviously \( E \) has \( \frac{n}{2} \) edges with label 0 and \( \frac{n-1}{2} \) edges with label 1.

That is, \( e_{f^*}(0) = \frac{n}{2} \) and \( e_{f^*}(1) = \frac{n-1}{2} \).
Therefore, \( |e_{f^*}(0) - e_{f^*}(1)| \leq 1 \) is satisfied.
Hence in both cases \( P_n \) admits a \( N \)-prime cordial labeling.

The \( N \)-prime cordial labeling of the graphs \( P_5 \) and \( P_8 \) are given in Example 6 and 7 respectively.

**Example 6.**

\[ 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \]
\[ 2^2 \quad 2^3 - 1 \quad 2^4 \quad 2^5 - 1 \quad 2^6 \]

Figure 1: \( N \)-prime cordial labeling of \( P_5 \)

**Example 7.**

\[ 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \]
\[ 2^2 \quad 2^3 - 1 \quad 2^4 \quad 2^5 - 1 \quad 2^6 \quad 2^7 - 1 \quad 2^8 \quad 2^9 - 1 \]

Figure 2: \( N \)-prime cordial labeling of \( P_8 \)

**Theorem 8.** The cycle \( C_n \) admits a \( N \)-prime cordial labeling.

**Proof.** Let \( C_n \) be the cycle of length \( n \).
Let \( V = \{v_i|1 \leq i \leq n\} \) be the vertex set and \( E = \{v_iv_{i+1}|1 \leq i \leq n-1\} \cup \{v_nv_1\} \)
be the edge set of the cycle \( C_n \).

**Case (1):** when \( n \equiv 1 \pmod{2} \)
Define an injective function \( f : V \to \mathbb{N}_0 \) such that
\[ f(v_i) = 2^{i+1}, \quad 1 \leq i \leq \frac{n+1}{2} \]
\[ f(v_i) = 2^{i+1} - 1, \quad \frac{n+1}{2} < i \leq n \]
and an induced edge function as in the Definition 2.
In this type of labeling pattern \( E \) has \( \frac{n+1}{2} \) edges with label 1 and \( \frac{n+1}{2} \) edges with label 0.
That is \( e_{f^*}(0) = \frac{n+1}{2} \) and \( e_{f^*}(1) = \frac{n-1}{2} \).
Therefore the condition \( |e_{f^*}(0)| - e_{f^*}(1)| \leq 1 \) is satisfied.

**Case (2):** when \( n \equiv 0 \pmod{2} \)
Define an injective function \( f : V \rightarrow \mathbb{N}_0 \) such that
\[
\begin{align*}
f(v_i) &= 2^i + 1; \quad 1 \leq i \leq \frac{n+1}{2} \\
f(v_i) &= 2^i + 1 - 1; \quad \frac{n+1}{2} < i \leq n
\end{align*}
\]
and an induced edge function as in the Definition 2.

In this type of labeling pattern \( E \) has \( \frac{n}{2} \) edges with label 0 and \( \frac{n}{2} \) edges with label 1.
That is, \( e_{f^*}(0) = \frac{n}{2} \) and \( e_{f^*}(1) = \frac{n}{2} \).
Therefore \( |e_{f^*}(0) - e_{f^*}(1)| \leq 1 \) is satisfied.
Hence in both cases the condition \( |e_{f^*}(0) - e_{f^*}(1)| \leq 1 \) is satisfied.

Hence \( C_n \) admits a \( N \)-prime cordial labeling.

The \( N \)-prime cordial labeling of \( C_6 \) is given in Example 9.

Example 9.

![Diagram](image)

\textbf{Figure 3:} \( N \)-prime cordial labeling of \( C_6 \)

\textbf{Theorem 10.} The helm graph \( H_n \) is a \( N \)-prime cordial graph.

\textbf{Proof.} Let \( H_n \) be the helm graph with \( 2n + 1 \) vertices and \( 3n \) edges.
Let \( V = \{v_0\} \cup V_1 \cup V_2 \) be the vertex set of \( H_n \) where \( V_1 = \{v_{i,1} \mid 1 \leq i \leq n\} \),
\( V_2 = \{v_{2,1} \mid 1 \leq i \leq n\} \).

Let \( E = E_1 \cup E_2 \cup E_3 \) be the edge set of \( H_n \) where \( E_1 = \{v_0v_{i,1} \mid 1 \leq i \leq n\} \),
\( E_2 = \{v_{i,1}v_{i+1,1} \mid 1 \leq i \leq n-1\} \cup \{v_{1,n}, v_{1,1}\} \) and \( E_3 = \{v_{i,2}v_{i+1,2} \mid 1 \leq i \leq n\} \).

\textbf{Case (1):} \( n \equiv 0 \pmod{2} \)

Define an injective function \( f : V \rightarrow \mathbb{N}_0 \) such that
\[
\begin{align*}
f(v_0) &= 1 \\
f(v_{1,i}) &= 2^{i+1}; \quad i \equiv 1 \pmod{2} \quad \text{and} \quad 1 \leq i \leq n \\
f(v_{2,i}) &= 2^{i+1} - 1; \quad i \equiv 0 \pmod{2} \quad \text{and} \quad 1 \leq i \leq n \\
f(v_{2,n}) &= 3^{i+1} - 1; \quad i \equiv 0 \pmod{2} \quad \text{and} \quad 1 \leq i \leq n
\end{align*}
\]

This type of labeling pattern with an induced edge function as in the Definition 2, the edge set \( E \) has \( \frac{3n}{2} \) edges with label 0 and \( \frac{3n}{2} \) edges with label 1.
That is, \( e_{f^*}(0) = e_{f^*}(1) = \frac{3n}{2} \). Hence \( |e_{f^*}(0) - e_{f^*}(1)| \leq 1 \) is satisfied.

\textbf{Case (2):} \( n \equiv 1 \pmod{2} \)

Define an injective function \( f : V \rightarrow \mathbb{N}_0 \) such that
\[
f(v_0) = 1
\]
This injective function with an induced edge function as in the Definition 2 gives the edge set \( E \) has \( \frac{3n-1}{2} \) edges with label 0 and \( \frac{3n+1}{2} \) edges with label 1.

That is, \( e_{f^*}(0) = \frac{3n-1}{2} \) and \( e_{f^*}(1) = \frac{3n+1}{2} \). Hence \( |e_{f^*}(0) - e_{f^*}(1)| \leq 1 \) is satisfied.

Hence the graph \( H_n \) is a \( N \)-prime cordial graph.

The \( N \)-prime cordial labeling of \( H_6 \) and \( H_5 \) are given in the following examples.

Example 11.
Example 12.

Figure 5: $N$-prime cordial labeling of $H_5$

4 Conclusion

In this paper we have introduced the new graph labeling known as $N$-prime cordial labeling of a graph and proved the existence of this labeling to the graphs such as path, cycle and helm graph.

References


