The Reciprocal S-Prime Meet Matrices on Posets

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Abstract

We consider Reciprocal S-Prime Meet Matrices on Posets as an abstract generalization of Reciprocal S-Prime greatest common divisor (Reciprocal S Prime GCD) matrices. We also found determinant and inverse and discuss the some of the most important properties of Reciprocal S-Prime GCD matrices are presented in terms of Meet Matrices.

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1 Introduction

Let \( S = \{x_1, x_2, x_3, \ldots, x_n\} \) be a set of n-positive integers with \( x_1 < x_2 < x_3 < \cdots < x_n \) and let \( f: p \rightarrow \mathbb{C} \) be a complex valued function on \( \mathbb{Z}_+ \) (i.e., arithmetical function). Let \( (x_i, x_j) \) denotes the greatest common divisor (GCD) of \( x_i \) and \( x_j \) and define the nxn matrices \( (S)_f = ((S)_f)_{i,j} = f(x_i, x_j) \). We refer to \( (S)_f \) as the GCD matrix on S with respect to f. The set \( S \) is said to be factor closed if it contains every positive divisor of each \( x_i \in S \) clearly a factor closed set is always GCD-closed and the converse does not hold. In 1876, the concept of classical Smith determinant with entries on \( \mathbb{Z}_+ \) was introduced by H.J.S. Smith [11] is,

\[
\det[(x_i, x_j)]_{n\times n} = \Phi(x_1) \Phi(x_2) \Phi(x_3) \ldots \Phi(x_n)
\]

\[
= \prod_{i=1}^{n} \Phi(x_i)
\]

where \( \phi \) is the Euler’s totient function. The GCD matrix with respect to \( f \) is,

\[
(f(x_i, x_j)) = \begin{bmatrix}
    f(x_1, x_1) & f(x_1, x_2) & \cdots & f(x_1, x_n) \\
    f(x_2, x_1) & f(x_2, x_2) & \cdots & f(x_2, x_n) \\
    \vdots & \vdots & \ddots & \vdots \\
    f(x_n, x_1) & f(x_n, x_2) & \cdots & f(x_n, x_n)
\end{bmatrix}
\]
and \( \det[f(x_i, x_j)] = \prod_{k=1}^{n} (f \ast \mu)(x_k) \). In [1992], S.Beslin and S.Ligh [5] generalized in this results on GCD matrices by showing that the determinant of the GCD Matrix on a GCD closed set \( S = \{x_1, x_2, x_3, \ldots, x_n\} \) is the product \( \prod_{k=1}^{n} (\alpha_k) \) where \( \alpha_k = \sum_{d|\{x_k, d|x_k\}} \Phi(d) \). Let \( S = \{x_1, x_2, x_3, \ldots, x_n\} \) be a set of distinct positive integers and the \( n \times n \) matrix \((S)_{ij} = (S_{ij})\) clearly \( (S_{ij}) = \frac{1}{4(x_i, x_j)+1} \), call it to be reciprocal S-prime GCD matrix on \( S \). If \( S \) is factor closed set then, \( \det_S = \prod_{i=1}^{n} g(x_i) \) where \( g(n) = \sum_{d|n} \frac{1}{d+1} (\mu(n/d)) \). In this paper describes an abstract generalization of Reciprocal S-prime GCD matrices, namely Reciprocal S-prime Meet Matrices on Posets. Some of the most important properties of Reciprocal S-prime GCD-matrices are presented in terms of Meet Matrices.Further, we found the determinant and inverse of the Reciprocal S-prime Meet Matrices.

2 Definition of Reciprocal S-Prime Meet Matrices

**Definition 1.** Let \((p, \prec) = (\mathbb{Z}^+, |)\) be a finite poset. We call \( P \) be a meet-semi lattice if for any \( x, y \in p \) there exist a unique \( z \in p \) such that (i) \( z \leq x \) and \( z \leq y \) and (ii) If \( w \leq x \) and \( w \leq y \) for some \( w \in p \) then \( w \leq z \). In such a case \( z \) is called the meet of \( x \) and \( y \) is denoted by \( x \land y \).

**Definition 2.** Let \( S \) be a subset of \( p \), we call \( S \) be a lower-closed if for every \( x, y \in p \) and \( x \in S \) and \( y \leq S \), we have \( y \in S \).

**Definition 3.** Let \( S \) be a subset of \( P \) then \( S \) is said to be meet-closed if for every \( x, y \in S \), we have \( x \land y \in S \). In this case \( S \) itself is a meet lattices. It is clear that a lower-closed subset of a meet semi-lattice is always meet-closed but not conversely. The concept “lower-closed” and “Meet-closed” are generalization of “factor-closed” and “GCD-closed” respectively. In what follows, let \( P \) always denotes a finite meet lattice, \( S \) a poset that can be embedded in a Meet-semi lattice and \( \bar{S} \) the unique minimal meet semi-lattice containing \( S \).

**Definition 4.** Let \( x \) and \( y \) be two elements the poset \( P \) and \( \mu \) is the mobius function of the poset \((S, \prec)\) then

\[
\mu(x, y) = \begin{cases} 
0 & \text{if } x \neq y \\
1 & \text{if } x = y \\
-\sum_{z: z \leq y} \mu(x, z) & \text{otherwise}
\end{cases}
\]

**Definition 5.** Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a subset of \( P \) the \( n \times n \) matrix \((s)_{ij} = ((s)_{ij})_{ij} = (f_{ij})\), where \( f_{ij} = \frac{1}{4(x_i, x_j)+1} \) is called the Reciprocal S-prime Meet Matrix on \( S \) with respect to \( f \).
3 Determinant of the Reciprocal S-Prime Meet Matrices

**Theorem 6.** Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a meet-closed subset of \( P \). Then 
\[
\det(S) = g(x_1) g(x_2) \ldots g(x_n),
\]
where \( g(x_i) \) defined by
\[
g(x_i) = \frac{1}{4 x_i + 1} - \sum_{x_j \leq x_i} g(x_j).
\]

**Corollary 7.** Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a lower-closed subset of \( P \). Then 
\[
\det(S) = g(x_1) g(x_2) \ldots g(x_n),
\]
where \( g(x_i) \) defined by
\[
g(x_i) = \sum_{x_j \leq x_i} \mu(x_j, x_i)
\]
or equality \( f(x_i) = \sum g(x_j) \), \( \mu \) being the Mobius function of \( P \).

**Example 8.** Let \( S = \{1, 2\} \) be a lower-closed set and \( (S, |) \) is a Poset. Then by using the Definition 2.5, we have the \( 2 \times 2 \) Reciprocal S-Prime Meet Matrix on \( S \).

\[
(S) = \begin{bmatrix}
\frac{1}{4(1 \wedge 1) + 1} & \frac{1}{4(1 \wedge 2) + 1} \\
\frac{1}{4(2 \wedge 1) + 1} & \frac{1}{4(2 \wedge 2) + 1}
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{5} & \frac{1}{5} \\
\frac{1}{5} & \frac{1}{5}
\end{bmatrix}
\]

\[
\det(S) = g(1) g(2) = g(1) g(2)
\]

By using the definition of \( g \) and \( \mu(x, y) \) we obtain:
\[
g(x_1) = g(1) = \sum_{x_j \subseteq x_1} \frac{1}{4 x_j + 1} \mu(x_j, 1)
\]
\[
= \frac{1}{4(1) + 1} \mu(1, 1)
\]
\[
= \frac{1}{5} (1) = \frac{1}{5}
\]
\[
g(x_2) = g(2) = \sum_{x_j \subseteq x_2} \frac{1}{4 x_j + 1} \mu(x_j, 2)
\]
\[
= \frac{1}{4(1) + 1} \mu(1, 2) + \frac{1}{4(2) + 1} \mu(2, 2)
\]
\[
= \frac{1}{5} (-1) + \frac{1}{9} (1) = -\frac{4}{45}
\]
\[
\det(S) = g(1) g(2) = \frac{1}{5} - \frac{4}{45}
\]
\[
= \frac{-4}{225}
\]

**Definition 9.** Let \( S = \{x_1, x_2, \ldots, x_n\} \) and \( T = \{y_1, y_2, \ldots, y_m\} \) be any two subsets of \( P \). Define the incidence matrix \( E(S, T) \) of \( S \) and \( T \) as an \( n \times m \) matrix whose \( i, j \) entry is 1 if \( y_j \leq x_i \) and zero otherwise namely. That is, \( E(S, T) = (e_{i,j})_{n \times m} \), where
\[
e_{i,j} = \begin{cases}
1, & \text{if } y_j \leq x_i \\
0, & \text{otherwise}
\end{cases}
\]

**Theorem 10.** Let \( S = \{x_1, x_2, \ldots, x_n\} \) be a subset of \( P \) with \( \bar{S} = \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+r}\} \). Let \( g \) be a function on \( \bar{S} \) defined as in Theorem 3.1. Then
\[
(S) = E \cdot \text{diag}(g(x_1), \ldots, g(x_{n+r})) \cdot E^T
\]
Theorem 11. Let $S, \bar{S}, f$ and $g$ be as in Corollary 3.2. Then
\[ \det(S) = \sum_{1 \leq k_1 < \cdots < k_n \leq n+r} \det(E_{(k_1, \ldots, k_n)})^2 g(x_{k_1}) \cdots g(x_{k_n}) \]
where $E_{(k_1, \ldots, k_n)}$ is the sub matrix of $E = E(S, \bar{S})$ consisting of the $k_1^{th}$, $k_2^{th}$, \ldots, $k_n^{th}$ columns of $E$. Furthermore; if $g$ is a function with positive values then
\[ \det(S) \geq g(x_1) g(x_2) \cdots g(x_n) \]
and the equality holds if and only if $S$ is meet-closed.

4 Inverse of Reciprocal S-Prime Meet Matrices

Theorem 12. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a lower-closed subset of $P$ and let $g(x_i) = \sum_{x_j \leq x_i} (4x_i + 1) \mu(x_j, x_i) \neq 0$ for all $x_j \in S$. Then $(S)$ is invertible and $(S)^{-1} = (b_{ij})$, where
\[ b_{ij} = \sum_{x_j \leq x_k, x_j \leq x_k} \frac{\mu(x_i, x_k) \mu(x_j, x_k)}{g(x_k)}. \]

Example 13. $(S)$ is a Reciprocal $S$-Prime Meet Matrix on lower-closed set $S = \{1, 2\}$. By Theorem 4.1, $(S)^{-1} = B = (b_{ij})$ where using Example 3.3.
\[ b_{11} = \sum_{1 \leq x_k} \frac{\mu(1, x_k) \mu(1, x_k)}{g(x_k)} \]
\[ = \frac{\mu(1, 1)^2}{g(1)} + \frac{\mu(1, 2)^2}{g(2)} \]
\[ = \frac{1^2}{5} + \frac{(-1)^2}{45} \]
\[ = 5 + \frac{1}{45} \]
\[ = \frac{20}{45} \]
\[ = \frac{4}{45} \]
\[ b_{11} = \frac{-25}{4} \]
\[ b_{12} = \sum_{1 \leq x_k} \frac{\mu(1, x_k) \mu(2, x_k)}{g(x_k)} \]
\[ = \frac{\mu(1, 2) \mu(2, 2)}{g(2)} \]
\[ = \frac{(-1)(1)}{4} \]
\[ b_{12} = \frac{45}{4} \]
Similarly,

\[ b_{21} = b_{12} = \frac{45}{4} \]
\[ b_{22} = \sum_{2|x_k} \frac{\mu(2, x_k) \mu(2, x_k)}{g(x_k)} \]
\[ = \frac{\mu(2, 2)^2}{g(2)} \]
\[ = \frac{1}{\frac{4}{45}} \]
\[ b_{22} = -\frac{45}{4} \]

Therefore,

\[ (S)^{-1} = B = \begin{bmatrix} \frac{-25}{4} & \frac{45}{4} \\ \frac{-25}{4} & \frac{45}{4} \end{bmatrix} \]

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References


