Complementary Nil Domination and Complementary Nil Eccentric Domination in Splitting Graph of a Graph

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Abstract

A subset D of the vertex set V(G) of a graph G is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an eccentric dominating set if for every v ∈ V−D, there exists at least one eccentric vertex of v in D. A set S ⊆ V is said to be a complementary nil dominating set of a graph G if it is a dominating set and its complement V−S is not a dominating set for G. The minimum cardinality of a cnd-set is called the complementary nil domination number of G and is denoted by γcnd(G). An eccentric dominating set D of G is a complementary nil eccentric dominating set if the induced subgraph ⟨V−D⟩ is not an eccentric dominating set for G. The minimum of the cardinalities of the complementary nil eccentric dominating set of G is called the complementary nil eccentric domination number γcned(G). For a graph G, let V′(G) = {v′ : v ∈ V(G)} be a copy of V(G). Then the splitting graph S_p(G) of G is the graph with the vertex set V(G) ∪ V′(G) and edge set {uv, u′v, u′v′ : uv ∈ E(G)}. In this paper, we obtain some bounds for γcnd(S_p(G)), γcned(S_p(G)). Exact values of γcnd(S_p(G)) and γcned(S_p(G)) for some particular classes of graphs are obtained.

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1 Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [5], Buckley and Harary [3].

Janakiraman, Bhanumathi and Muthammai [7] introduced Eccentric domination in Graphs. Splitting graphs were first studied by Sampathkumar and Walikar [9] and were further developed by Patil and Thangamari [8]. Janakiraman, Muthammai and Bhanumathi studied eccentricity properties of splitting graphs in [6]. Tamil Chelvamand Robinson Chellathurai studied the concept of complementary nil eccentric domination number [5]. Bhanumathi and Sudhasenthil studied the concept of complementary nil eccentric domination in splitting graphs of some graphs [2].

**Definition 1.1:**
Let G be a connected graph and v be a vertex of G. The eccentricity $e(v)$ of v is the distance to a vertex farthest from v. Thus, $e(v) = \max\{d(u, v) : u \in V\}$. The radius $r(G)$ is the minimum eccentricity of the vertices, whereas the diameter $\text{diam}(G) = d(G)$ is the maximum eccentricity. For any connected graph G, $r(G) \leq \text{diam}(G) \leq 2r(G)$. The vertex v is a central vertex if $e(v) = r(G)$. The center $C(G)$ is the set of all central vertices. For a vertex v, each vertex at a distance $e(v)$ from v is an eccentric vertex of v. Eccentric set of a vertex v is defined as $E(v) = \{u \in V(G) / d(u, v) = e(v)\}$.

**Definition 1.2:** A set $D \subseteq V$ is said to be a dominating set in G, if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number and is denoted by $\gamma(G)$ [3, 11].

**Definition 1.3:** A dominating set $D$ of a graph G is a split dominating set if the induced subgraph $\langle V - D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ is the minimum cardinality of a split dominating set.

**Definition 1.4:** Let $S \subseteq V$. Then a vertex $v \in S$ is said to be an enclave of S if $N[v] \subseteq S$.

**Definition 1.5:** A set $D \subseteq V(G)$ is an eccentric dominating set if D is a dominating set of G and for every $v \in V - D$, there exists at least one eccentric vertex of v in D. The minimum cardinality of an eccentric dominating set is called the eccentric domination number and is denoted by $\gamma_{ed}(G)$. If D is an eccentric dominating set, then every superset $D' \supseteq D$ is also an eccentric dominating set. But $D' \subseteq D$ is not necessarily an eccentric dominating set. An eccentric dominating set D is a minimal eccentric dominating set if no proper subset $D'' \subseteq D$ is an eccentric dominating set [5].

**Definition 1.6:** A set $S \subseteq V$ is said to be a complementary nil dominating set (cnd-set) of a graph G if it is a dominating set and its complement $V - S$ is not a dominating set for G. The minimum cardinality of a cnd-set is called the complementary nil domination number of G and is denoted by $\gamma_{cnd}(G)$.

**Definition 1.7:** An eccentric dominating set D of G is a complementary nil eccentric dominating set(cned-set) if the induced subgraph $\langle V - D \rangle$ is not an eccentric dominating set for G. The minimum of the cardinalities of the complementary nil eccentric dominating set of G is called the complementary nil eccentric domination number $\gamma_{cned}(G)$. 
**Definition 1.8:** For a graph $G$, let $V'(G) = \{v' : v \in V(G)\}$ be a copy of $V(G)$. Then the splitting graph $S_p(G)$ of $G$ is the graph with the vertex set $V(G) \cup V'(G)$ and edge set \{uv, u'v' : uv \in E(G)\}. Clearly, (i) For any graph $G$, $\gamma(G) \leq \gamma_{ed}(G)$.(ii) For any graph $G$, $\gamma(S_p(G)) \leq \gamma_{ed}(S_p(G))$.

**Theorem 1.** For any graph $G$, $p/(1+\Delta(G)) \leq \gamma(G) \leq p-\Delta(G)$.

**Theorem 2.** $r(S_p(G)) = \max\{2, r(G)\}$.

**Theorem 3.** If $G$ is of radius two with a unique central vertex $u$, then $\gamma_{cned}(G) \leq p-\deg(u)$.

**Theorem 4.** Let $G$ be a graph with rad$(G) \geq 3$, then $\gamma_{cned}(G) \leq p-\delta(G)$.

### 2 Complementary Nil Domination and Complementary Nil Eccentric Domination in Splitting Graphs

In this section, we study the complementary nil domination number and complementary nil eccentric domination number of $S_p(G)$ for some graph $G$.

**Example:**

$D_1 = \{v_3, v_4\}$ is a dominating set of $G$, $\gamma(G) = 2$.

$D_2 = \{v_1, v_5, v_6\}$ is an eccentric dominating set of $G$ and also a complementary nil eccentric dominating set of $G$. $\gamma_{ed}(G) = \gamma_{cned}(G) = 3$.

$D_3 = \{v_3, v_4, v_5\}$ is a complementary nil dominating set of $G$. $\gamma_{cnd}(G) = 3$.

$S_1 = \{v_3, v_4\}$ is a dominating set of $S_p(G)$, $\gamma(S_p(G)) = 2$.

$S_2 = \{v_1, v_3, v_4, v_5, v_t\}$ is an eccentric dominating set of $S_p(G)$ and also a complementary nil eccentric dominating set of $S_p(G)$. $\gamma_{ed}(S_p(G)) = \gamma_{cned}(S_p(G)) = 5$.

$S_3 = \{v_3, v_4, v_5, v_t\}$ is a complementary nil dominating set of $S_p(G)$. $\gamma_{cnd}(S_p(G)) = 3$.

Here, $\gamma_{ed}(G) = \gamma_{cnd}(G) = \gamma_{cned}(G) = \gamma_{cnd}(S_p(G)) = 3$. 

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Proposition:

1. (i) $\gamma_{cnd}(Sp(K_1,p)) = 3$
2. $\gamma_{cned}(Sp(K_1,p)) = \begin{cases} 2 & \text{if } p = 1, \\ 3 & \text{if } p > 1. \end{cases}$
3. $\gamma_{cnd}(Sp(K_p)) = \gamma_{cned}(Sp(K_p)) = p+1$, $p > 2$. $\gamma_{cnd}(Sp(K_2)) = 3$, $\gamma_{cned}(Sp(K_2)) = 2$.
4. (i) $\gamma_{cnd}(Sp(W_p)) = p+1$.
   (ii) $\gamma_{cned}(Sp(W_p)) = 4$, $p > 3$. $\gamma_{cned}(Sp(W_3)) = 3$.

Lemma: 2.1 Let $S$ be a cnd-set of a splitting graph $Sp(G)$. Then $S$ contains at least one enclave of $S$.

Proof. Let $S$ be cnd-set of a splitting graph $Sp(G)$. By the definition of cnd-set, $V-S$ is not a dominating set, which implies that there exists a vertex $v \in S$ such that $v$ is not adjacent to any vertex in $V-S$ and so $N[v] \subseteq S$. That is $S$ contains at least one enclave of $S$.

Proposition: Let $G$ be a graph and $S$ be a $\gamma_{cnd}$-set of $Sp(G)$. If $u$ and $v$ are enclaves of $S$, then $N[u] \cap N[v] \neq \phi$ and $u$ and $v$ are adjacent.

Proof. Let $u$ and $v$ be two enclaves of $S$. Suppose $N[u] \cap N[v] = \phi$. Then $u$ is an enclave of $S-N(v)$. Clearly $S-N(v)$ is a cnd-set of $Sp(G)$ and $S-N(v) \subseteq S=\gamma_{cnd}(Sp(G))$, which is a contradiction to the minimality of $S$. Hence $N[v] \cap N[v] \neq \phi$. Suppose $u$ and $v$ are non-adjacent, $u \notin N(v)$ and so $S-\{v\}$ contains an enclave $u$ of $S-\{v\}$. Hence, $S-\{v\}$ is a cnd-set, which is a contradiction to the minimality of $S$.

Theorem 5. For any splitting graph $Sp(G)$, every $\gamma_{cnd}$-set of $Sp(G)$ intersects with every $\gamma$-set of $Sp(G)$.

Proof. Let $S_1$ be a $\gamma_{cnd}$-set of $Sp(G)$ and $S$ be a $\gamma$-set of $Sp(G)$. Suppose that $S_1 \cap S = \phi$ then $S \subseteq V(Sp(G))-S_1$, $V(Sp(G))-S_1$ contains a dominating set $S$. Therefore, $V-S_1$ itself is a dominating set of $Sp(G)$, which is a contradiction. Thus, $S_1 \cap S \neq \phi$.

Theorem 6. For any graph $G$, $\delta(G)+1 \leq \gamma_{cnd}(Sp(G)) \leq \gamma(Sp(G))+\delta(G)$.

Proof. Let $S$ be a $\gamma_{cnd}$-set of a splitting graph $Sp(G)$. Since $V-S$ is not a dominating set of a splitting graph $Sp(G)$, there exists a vertex $v \in S$ which is not adjacent to any of the vertices in $V-S$. Therefore, $N[v] \subseteq S$ which implies that $N(v) \subseteq S$, that is $d(v)+1 \leq S$ and so $\delta(Sp(G))+1 \leq \gamma_{cnd}(Sp(G))$. Since $\delta(Sp(G)) = \delta(G)$. Therefore, $\delta(G)+1 \leq \gamma_{cnd}(Sp(G))$. Let $S_1$ be a $\gamma$-set of $Sp(G)$. Let $u \in V$ such that $d(u) = \delta$. Then at least one vertex $u_1 \in N[u]$ such that $u_1 \in S_1$. Now $S_1 \cup (N[u]-\{u_1\})$ is a cnd-set of $Sp(G)$, which implies that $\gamma_{cnd}(S_1+\{u_1\}) \leq S_1+\{u_1\} \leq S_1+N[u]-\{u_1\} = \gamma(Sp(G))+\delta(G)$. Therefore, $\delta(G)+1 \leq \gamma_{cnd}(Sp(G)) \leq \gamma(Sp(G))+\delta(G)$.
Corollary 7. For any graph $G$ with $\delta(G) = 1$, $\gamma_{cnd}(S_p(G)) = \gamma(S_p(G)) + 1$.

Proposition For any graph $G$ $\gamma_s(S_p(G)) < \gamma_{cnd}(S_p(G))$.

Proof. Let $S$ be a $\gamma_{cnd}$-set of $S_p(G)$. Then there exists a vertex $v \in S$ such that $N[v] \subseteq S$. Clearly $S - \{v\}$ is a split dominating set of $S_p(G)$ and so $S - \{v\} \geq \gamma_s(S_p(G))$. That is, $\gamma_{cnd}(S_p(G)) - 1 \geq \gamma_s(S_p(G))$. Therefore, $\gamma_s(S_p(G)) \leq \gamma_{cnd}(S_p(G)) - 1 < \gamma_{cnd}(S_p(G))$. \hfill $\square$

Theorem 8. Let $S$ be a complementary nil eccentric dominating set of a splitting graph $S_p(G)$. Then $S$ contains at least one enclave of $S$ or $S$ contains at least one vertex whose eccentric vertices are in $S$.

Proof. Let $S$ be a $\gamma_{cnd}$-set of a splitting graph $S_p(G)$. By the definition of $\gamma_{cnd}$-set, $V - S$ is not an eccentric dominating set, which implies that there exists a vertex $v \in S$ such that $v$ has no eccentric vertices in $V - S$. Therefore, $v$ has all its eccentric vertices in $S$ or there exists a vertex $v \in S$ such that $v$ is not adjacent to any of the vertices in $V - S$. That is $N[v] \subseteq S$. That is $S$ contains at least one enclave of $S$. \hfill $\square$

Theorem 9. For any graph $G$, every $\gamma_{cnd}$-set of $S_p(G)$ intersects with every $\gamma_{ed}$-set of $S_p(G)$.

Proof. Let $S$ be a $\gamma_{cnd}$-set of $S_p(G)$ and $S$ be a $\gamma_{ed}$-set of $S_p(G)$. Suppose that $S \cap S = \emptyset$ then $S \subseteq V(S_p(G)) - S_1$, $V(S_p(G)) - S_1$ contains eccentric dominating set $S$. Therefore, $V - S_1$ itself is an eccentric dominating set of $S_p(G)$, which is a contradiction. Thus, $S \cap S \neq \emptyset$. \hfill $\square$

Theorem 10. If $G$ is a graph of radius two with a unique central vertex then $\gamma_{cnd}(S_p(G)) \leq 2(p - \deg_G(u))$.

Proof. By Theorem 1.2, $S_p(G)$ is of radius two and has a unique central vertex. If $u$ is a central vertex of $G$ then $u$ is a central vertex of $S_p(G)$ and $\deg u$ in $S_p(G)$ $= 2 \deg u$ in $G$. Therefore, By Theorem 1.3, $\gamma_{cnd}(S_p(G)) \leq 2p - 2 \deg_G(u) \leq 2(2 - \deg_G(u))$. \hfill $\square$

Theorem 11. If $G$ is a graph with a pendant vertex, then $\gamma_{cnd}(S_p(G)) \leq 2\gamma_{ed}(G)$ or $2\gamma_{ed}(G) - 1$.

Proof. Let $D$ be a $\gamma_{ed}$-set of $S_p(G)$. If $u$ is a pendant vertex in $G$ then $u$ is a pendant vertex in $S_p(G)$. If $u$ and its support vertex is in $D$, then $V - D$ is not a dominating set. Therefore, $\gamma_{cnd}(S_p(G)) = 2\gamma_{ed}(G)$. If $u$ or its support vertex $v$ is not in $D$, then $D_1 = D - \{u\}$ (or) $D_1 = D - \{v\}$ is an eccentric dominating set and $V - D_1$ is not a dominating set. Therefore, $\gamma_{cnd}(S_p(G)) = 2\gamma_{ed}(G) - 1$. \hfill $\square$

Theorem 12. For any graph $G$, $\gamma_{ed}(S_p(G)) \leq \gamma_{cnd}(S_p(G)) \leq \gamma(S_p(G)) + t$, where $t$ is the number of eccentric vertices of $S_p(G)$.

Proof. Obviously $\gamma_{ed}(S_p(G)) \leq \gamma_{cnd}(S_p(G))$. Let $D$ be a minimum dominating set of $S_p(G)$. Let $S = \{u \in V(S_p(G)) / u$ is an eccentric vertex of some $v \in V(S_p(G))\}$. Then clearly $D \cup S$ is an eccentric dominating set. Also $V(S_p(G)) - (D \cup S)$ has no
eccentric vertices. So $D \cup S$ is a complementary nil eccentric dominating set. Hence, 
$\gamma_{cned}(S_p(G)) \leq D+S= \gamma(S_p(G)) + t$. 

**Theorem 13.** Let $G$ be a graph with $\text{rad}(G) \geq 3$, then $\gamma_{cned}(S_p(G)) \leq 2(p-\delta(G))$.

*Proof.* By Theorem 1.2, $S_p(G)$ is of radius greater than two. By Theorem 1.4, $\gamma_{cned}(S_p(G)) \leq 2(p-\delta(S_p(G)))$. Since $\delta(S_p(G)) = \delta(G)$. Therefore, $\gamma_{cned}(S_p(G)) \leq 2(p-\delta(G))$. 

**Theorem 14.** For any graph $G$, $\gamma_{cned}(S_p(G)) \leq 2\gamma_{ed}(G) + \delta(G)$.

*Proof.* Let $S$ be the $\gamma_{ed}$-set of $G$ and $u \in V(S_p(G))$ such that $d(u) = \delta(G)$. If $u \in V(S_p(G)) - S$, there exists $v \in N(u)$ such that $v \in S$, $N(u) = \delta(G)$. Now $S_1 = S \cup N[u]$ is a $\gamma_{ed}(G)$-set and $V - S_1$ is not a dominating set. Hence, $\gamma_{cned}(S_p(G)) \leq S+N[u]= 2\gamma_{ed}(G) + \delta(G)$. 

### 3 Conclusion

Here we have studied complementary nil domination and complementary nil eccentric domination in splitting graph of some families of graph and also studied some bounds for the complementary nil domination and complementary nil eccentric domination in splitting graph of a graph.

### References


